Teaching and Learning Complex Analysis at Two British Columbia Universities

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Abstract

The objective of the present study was to investigate teaching and learning in typical university level complex analysis classes in British Columbia. Three classes were studied at two universities. Presentations of material in class by instructors was carefully recorded. A total of 20 subjects were studied during 54 audiotaped interviews of one hour duration. During interviews students worked on 6 to 10 questions, recording their work on worksheets. Transcripts were made of the interviews and these together with the worksheets were analyzed to study student understanding of the many concepts covered in a beginning complex analysis course.

In this thesis we have confined our analysis to the data we collected on just one theme: multirepresentations of complex numbers. We found that students have many misconceptions and difficulties with the basic representations of complex numbers such as,
$z = x + iy$, $z = (x, y)$, $z = re^{i\theta}$, and the symbolic representation, in which z is used directly. In addition, we studied how well students were able to judge when to shift from one representation to another. Finally, we have examined the data we collected that shows how insufficient understanding of basic material affects student ability to do problems from material covered later in the course.

Where possible we have attempted to identify different levels of understanding of the various representations. Although we found very little Mathematical Education literature on complex analysis, our analysis of our data supports results reported in the literature on representations of fractions and more general representations.

We found that students were competent with the $z = x + iy$ and $z = re^{i\theta}$ representations, and most made reasonable shifting decisions between these two forms, but there is little evidence that students understood the symbolic representation of complex numbers. In any case, from our data we have identified four characteristics of understanding a given representation that are consistent with results reported in the literature.

We have also studied how well students were able to understand the $z = (x, y)$ representation simply as a mathematical form with certain rules. Four stages of understanding of this question were identified.

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TABLE OF CONTENTS

List of Tables, Figures and Boxes

Chapter 1

Nature and Goals of this Study

1.1 Introduction

We were motivated to do a mathematics educational study of complex analysis by our mutual loves of complex analysis and teaching. Complex analysis is one of the most beautiful fields in all of mathematics; practically no one who studies this subject does so without being struck by the wonderful interplay between ideas from calculus, algebra, and geometry. In British Columbia, complex analysis is typically a third year course, with subsequent fourth year and graduate courses sometimes available. The third year version of the course is usually oriented towards science and engineering students, with a strong focus on applications.

1.2 Research Rational

Aside from our great personal love for the subject of complex analysis, we expected the subject to be an excellent area of research in post-calculus Mathematics Education for the following reasons:

1.2.1 Intellectual Inquiry

1. The topics covered in third year complex analysis include a variety of subjects. Some of these topics, depend on a thorough development of simpler concepts, but others do not. We conjectured that this aspect of complex analysis would enable us to study basic material for its own sake, as well as enable us to study the effect that misunderstandings of basic material has on later work. In addition, we expected to be

able to study some advanced topics independently of the extent to which students understand basic material. For example, we expected to study topics such as singularities or rational functions, which are usually studied well into the course, but are not especially dependent on earlier work.

2. We believed at the time we began this research, that research in Mathematics Education at third year level and beyond was useful because we believed that students in programs that required a course in complex analysis were quite likely to use what they have learned, so that the effort to learn how to teach them better would have extra value. This contrasts with, for example, first year calculus courses in which many students are enrolled because the program they wish to take is using calculus as an entrance screen.

3. We expected to encounter three types of problems that students may not have confronted in lower level courses. For example, at Simon Fraser University students do not generally have to construct proofs until third year (the standard second year analysis course is not a prerequisite for complex analysis at either the universities studied). In addition, success at complex analysis requires a greater synthesis of concepts such as algebra, trigonometry, calculus, geometry, topology, and symbolism than other courses at third year level. Finally, the shift from thinking "real" to thinking "complex" is not trivial.

1.2.2 Feasibility

1. Complex analysis is rich in concepts, both difficult and relatively straight forward. We conjectured that the variation of difficulty of the concepts, would enable us to include students with a wide range of capabilities.

2. Since complex analysis involves concepts from calculus, algebra, geometry and topology, we expected that we would have many opportunities to study how well students synthesized concepts from different branches of mathematics.

3. Enrollment is high enough, and course offerings frequent enough in third year mathematics courses at Simon Fraser University and the University of Bristish Columbia, to attract acceptable samples of students for study.

1.3 Research Objectives

With the above considerations in mind we were able to formulate general objectives for our research. The first objective was to investigate what students are learning in third year complex analysis courses at British Columbia universities. We intended to fulfill this objective on more than one level. To begin with, we planned to simply catalogue what happens in a third year complex analysis course. This included primarily course content and presentation, but as well ethnographic questions, such as, student responses to what was happening in class.

Next, we proposed to survey what sort of problems students have with the concepts introduced in the classes studied. We were particularly interested in determining which topics might be made more accessible to students, with better or different instruction.

Finally, we planned to begin the process of identifying those topics and (learning difficulties) that would be fruitful for further research. We intended to begin a detailed study of one or two of these identified topics, if time permitted, in order to get an impression of how our future research of these topics would transpire.

In addition to the above objectives, we were also interested in some specific issues:

1. How do misunderstandings of the basic material affect student conceptions about later material?

2. Is there any evidence of students making the transition from a *process* understanding to an *object* understanding of a given mathematical concept (*process* and *object* understandings are discussed at length in section 2.1.2.5)?

3. What could we learn about how students are synthesizing the different branches of mathematics required to understand complex analysis?

1.4 Overview

With the above objectives and considerations in mind we surveyed the literature, first for research reports in complex analysis and then for research reports on calculus and algebra that we could adapt to our purposes. We also established a research plan and methodology. In chapter 2 we report on the theoretical framework that we used for this study.

In chapter 3 we report on our literature review. Since we found almost no papers on the subject of complex analysis from the perspective of mathematics education, we have reported on a number of studies done in other fields of university level mathematics, such as algebra (both abstract and linear) and calculus, that we used as models for specific questions on questionnaires, general ideas for interviewing, ideas for analysis of data, etc.

Chapter 4 includes a thorough discussion of the epistemology of the complex numbers. In addition, we have traced the history of complex numbers in chapter 4.

In chapter 5 we present our methodology. This chapter includes both how we collected our data, and the framework we chose to analyze our data.

In chapter 6 we report on our data and present our analysis of that data. We have organized our results into the general heading of Multirepresentations of Complex Numbers.

Chapter 7 consists of conclusions, including suggestions for possible future directions for research in this field.

Chapter 2

Theoretical Framework

2.1 Research Model

We began our research preparation by identifying a research model. We were interested in three areas: the teaching model used, a theory of learning, and a theory of mathematical knowledge.

2.1.1 Teaching Model

In this study we had no control over the teaching model used, however, we expected it to be essentially the Old Humanist style described in Ernest [1] (we have chosen not to analyze the data that we collected on the teaching model, but generally, of the five teaching models described in the Ernest scheme, the Old Humanist model is the best description of what actually happened).

In the Old Humanist model the emphasis of teaching is conveying a body of knowledge. Mathematical knowledge is assumed to be absolute, and essentially independent of the people and situations of those who study it. It is up to the student to learn the material presented. The teacher is an expert in the field (certainly the case in all three of the classes studied). For the Old Humanist school, teaching consists primarily of lecturing and answering questions. Mathematical knowledge is determined by standards set by the community of publishing mathematicians. Accordingly, learning difficulties of individual students, differences between cultures, etc. are considered largely irrelevant to teaching, although individual teachers may have varying degrees of empathy for these difficulties.

2.1.2 Learning Model

We based our study of how students are learning complex analysis on the general model developed by Confrey [2, 3, 4]. However, before we discuss Confrey's model we will review the representation model, constructivism and radical constructivism, since Confrey's model is a response to some of the criticisms of radical constructivism.

The representational model essentially says that the learner builds an internal representation of a mathematical concept by having a transparent external representation made to them by a teacher. The teacher may have constructed the *transparent external representation*, but generally this is done by an expert in the field via the course materials.

Constructivism began to be adopted by the mathematics education community as a replacement for the the representational model in the 1980's and 1990's as contradictions in this model became apparent. An additional impetus for new theories of learning was a growing dissatisfaction (as evidenced by, for example, the calculus reform movement) with the results of teaching by the representational view. Thus, the primary roots of constructivism within the mathematics education community were dissatisfaction with teaching methods and educational philosophy.

Radical constructivsim on the other hand, developed out of dissatisfaction with existing models of scientific knowledge. The "reform" movement in the philosophy of science began in the 1960's, as many philosphers began to challenge the view that scientific inquiry was fuelled by the scientific method. There was a parallel reform movement in psychology that also led to radical constructivisim, but we will focus exclusively on the mathematics/ mathematics educational origins, even though the two movements were interrelated. For an account of the origins of radical constructivism in the field of psychology see Steffe and Kieren [5].

In any case, as the reform movement in philosophy of science progressed, researchers in the field of mathematics education used the work of Lakatos and Piaget to come to the view that the theory of constructivism did not go far enough. (Incidently, the work of Piaget was also pivotal to the development of radical constructivism in the field of psychology.) To address the new concerns, the additional postulates of radical constructivsim were developed. (There are several versions of radical constructivism, but all are related to constructivism).

Thus, we begin our discussion of Confrey's model with a summary of the representational view, constructivism and radical constructivism and the relationships between these theories.

2.1.2.1.1 The Representational Model

The representational model of learning was the dominant learning model used in mathematics education until constructivist models began to be adopted in the 1980's. Rorty [6] has described the main theme of the representational view of the mind as:

> To know is to represent accurately what is outside the mind; so to understand the possibility and nature of knowledge is to understand the way in which the mind is able to construct such [internal] representations. (p. 3)

Thus, according to the representational view, learning a mathematical concept consists of constructing, inside our heads, an accurate representation of the mathematical concept.

The representational model was (and still is) attractive because the model is compatible with three components of instruction that are widely practiced:

> 1. The primary purpose of instruction is to help students construct mental representations that correctly depict mathematical relationships located outside the mind of the student in the instructional materials or the mind of the instructor.
>
> 2. The main method of achieving instructional goals is to develop *transparent* instructional materials that make it possible for students to form the correct internal representations of the mathematical concepts under study.
>
> 3. The external instructional materials together with the teacher's presentation of those materials are the foundation from which students build their mental representations of the mathematics under study, and hence, their mathematical knowledge.

For an example of this approach within the context of a lesson on complex numbers, suppose the topic is addition and/or multiplication of complex numbers using the algebraic extension representation, $z = x + iy$. The overall goal of the lesson would be to help students construct a representation in their minds of addition and/or multiplication of complex numbers. Of course, the complexity of the mental representation will depend on the depth of the presentation. One could use as their *transparent instructional representation*, the properties of the real numbers, and treat $(a + ib)(c + id)$ as though it were a problem in real number multiplication, with the additional special rule that $i^2 = -1$. In this case, students are assumed to be familiar with the properties of the real numbers. External instructional materials would consist of any one of numerous excellent textbooks on the subject of complex analysis.

It should be clear from this example that the representational view is consistent with actual classroom practice in a wide variety of settings, irrespective of whatever the instructors of these classes profess to believe. Nevertheless, the representational view has come under increasing criticism in the last two decades.

2.1.2.1.2 Criticism of the Representational Model

Cobb, Yackel and Wood [7] have classified the objections to the representational view into four categories: theoretical, anthropological, pedagogical, and philosophical (duality).

Theoretical Objections to the Representational Model

The theoretical argument against the representational view goes like this: If the representation view is valid, then students will *inevitably* construct a representation of the mathematics under study that accurately reflects the external concepts. This means that learning is triggered by the mathematical concepts under study. But by assumption, the student has no understanding of the new concept, so how can a concept of which the student has no mental representations trigger construction of those representations?

One answer to this objection is that the teacher facilitates the process. However, this answer is inadequate for several reasons: 1. New mathematics is invented (discovered) all the time, so one can further one's mathematical understanding entirely by reflection. 2. Despite the best efforts of teachers everywhere, students routinely construct representations that are incorrect (by the representational view), suggesting that there is more to learning than *transparent* presentation by the teacher. 3. Steinbring [8] and Brousseau [9] have both noted that the more detailed the instruction, the more student understanding seems to be lost (in this context "detailed" does not necessarily mean careful or thoughtful instruction). In other words, they found that a very detailed *transparent presentation* does not necessarily lead to good understanding. 4. Finally, the trend in mathematics education is towards teaching as a process of negotiation (Bishop, [10]) as opposed to a process of imposition. This trend is reflected, for example in the calculus reform movement. Thus, the representational view of learning faces a serious theoretical challenge.

Anthropological Objections to the Representational Model

The anthropological objections to the representational model arise from close scrutiny of what is meant by the *experienced transparency of external representations for the expert*. There are essentailly two questions that arise from close examination of the experience of the expert: How can we expect the student, who after all, is assumed to be uninitiated into the culture of expert mathematicians, to understand anything about what the instructor is presenting to the class? In this question we are not even considering cultural differences. The question is how can we expect a student to understand anything about the presentation by the instructor, even assuming they speak the same language, live in the same culture, social class, are of the same gender, etc.? No matter how clear or transparent the presentation is, there are a multitude of relationships and connections that students *could* and do make from the lesson. So how can we claim that the presentation was *transparent*? The second question is related to the first but the question is how can we ignore cultural differences and differences of affect when defining the transparent experience of the expert?

Cobb, Yackel and Wood [7] argue that the ability of teachers and students to overcome apparently insurmountable psychological and cultural differences indicates that students must be actively constructing their representations as opposed to absorbing them from the instructor. Thus, teaching is an act of guiding the student's constructive efforts. In other words, learning is an interactive process that involves negotiation between the learner and teacher. While the representational model of learner could perhaps be modified to accommodate the view advocated by Cobb, Yackel and Wood (and many other cited therein), the model of teacher and student negotiating acceptable mental representations of mathematical concepts for the student is not consistent with the characteristics of the representation model given in section 2.1.2.1.1. Thus, anthropological and psychological considerations lead one to question the representational model

Pedagogical Objections to the Representational Model

Cobb, Yackel and Wood [7] (and several authors cited therein) argue that the representational learning model, if rigidly applied, can lead to students separating mathematical activity in school and outside of school. The problem is that the presentation of materials *transparent to the expert* is clearly quite a different experience for students than the mathematizing that students do outside of school. This means that for many students school mathematics remains irrelvent and out of reach, since they cannot connect the concepts they are learning in school with the math they are learning outside of the class room.

Philosophical Objections to the Representational Model

If we look closely at the representational learning model, it is evident that there is a duality: on the one hand is the mathematics to be learned (external representation) and on the other hand is the mathematics in the student's head (internal representation). Thus, the internal and external representations are fundamentally separated, and the basic problem of learning and (teaching) is how to bring the internal and external together again.

Von Glasersfeld [11] has reviewed the many unsuccessful attempts by philosophers to solve the problem of uniting the mind (internal) with the external. The implication is that if the separation of mind and environment does not have a solution in other (philosophical) settings then it is unlikely that there is a fully satisfactory solution within the representational model.

More recently, many philosophers, such as Gadamer [12], Habermas [13], Bernstein [14, 15], Putnam [16] and Rorty [6, 17] have challenged the underlying assumption of the problem, namely the idea that the mind builds representations of reality independent of history, social factors, and human requirements. Thus, the problem of the duality of the mind and the external environment (posed by Descartes) is rejected as an illusion based on a faulty premise. These philosphers, consequently, reject representational models of learning: for example, Dewey [18] has referred to representational models of learning as *spectator theories of knowledge* to focus attention on the fact that the representational models of learning ignore environmental and cultural factors, as well as the collective activities of humans.

Another way that the same ideas have been expressed (for example Searle, [19]), is that the only mind we will ever know is the one that is involved in all aspects of our lives, so why bother postulating the existence of the internal mind of representational models?

In any case, we can see that supporters of the representational model at the very least have a serious philosophical challenge to address if the credibility of the representational model is to be retained.

2.1.2.1.3 Summary of the Representational Model of Learning

The representational model of learning mathematics is the view that students construct accurate mental representations of mathematics presented in terms of models and images that are obvious to an expert. The representational model has been criticized on several grounds, namely theoretical, anthropological and psychological, pedagogical, and philisophical.

From a theoretical point of view, there is the question of how a student can be expected to construct new representations if these representations are required to be already present in order to trigger the development of the new representations?

From a psychological point of view, we can wonder how a student picks out the correct relationship about which to form a mental representation, when even in the simplest situation there are a multitude of possible relationships from which to choose?

From an anthropological point of view, we can ask whether or not an expert mathematician can ever be expected to present material in a way that is *transparent* to all students from many different backgrounds and experience.

The pedagogical objection to the representational model, that we have discussed, is that by focussing on the transparent representations of experts, school math is often very different from math outside of school.

Finally, many philosophers have rejected representational models on the grounds that the notion of mind independent of human activity is unfounded and unprovable, and therefore not useful.

Having discussed the representational model at some length, we are now ready to proceed to the first attempts by contructivists to address the shortcomings of the representational models.

2.1.2.2 Constructivism

Constructivism arose in response to the shortcomings of the representational model, as well as in response to the need for a philosophy of learning that was consistent with the increasing trend to negotiation as opposed to imposition of knowledge, Bishop [10]. In this section we discuss constructivisim, and some of the critiques of constructivism. We are particularly interested in critiques of constructivism that have lead many researchers in mathematics educations to adopt the perspective of radical constructivism, which is the topic of section 2.1.2.3.

2.1.2.2.1 Constructivism - The Basic Model

The central feature of constructivism is the recognition that the learner actively constructs their own understanding. This has been described in various ways:

> Knowledge is actively constructed by the cognizing subject, not passively received from the environment (Kilpatrick, [20]).
>
> These days the constructivist perspective typically stands for a gentle commitment to mentalism - a commitment to the belief that mental structures exist, that such structures shape the ways individuals see the world, and that people build those structures through interactions with the world around them (Schoenfeld, [21, p. 290]).
>
> Whereas, the term "constructivism" was intended to convey the Piagetian notion of the nature of building cognitive structures, for many people it has come to mean a way of teaching embodied by one of two classroom methods. For some, having students use manipulatives is the necessary and sufficient condition for "doing constructivist teaching". ... The other quality of the instructional environment often thought to be *per se* constructivist is the use of group discussion (Pirie and Kieren, [22, p. 505]).

Thus, constructivism is an attempt to revise the representational model by recognizing that the learner actively builds their own knowledge of the subject under study.

The problem faced by the constructivist is the problem of coordinating three aspects of the learning process: the assumed to be shared mathematical culture of society, the assumed to be shared culture of the classroom, and the thinking and learning patterns of the student. This problem contrasts with the dilemma faced by the representationalist, namely how to reconcile the duality between the student and the expert.

Perhaps it is apparent from the discussion so far that presenting a "model" constructivist learning scenario is not easy, since exactly what constitutes a constructivist learning environment is still not clear. Thus, we will be content with a few ideas of what constructivist teaching and learning should look like. It should be noted that even though this section is part of our development of a learning model, so that matters of teaching should properly go in section 2.1.1, in the literature constructivist teaching and learning have consistently been discussed hand in hand. To us treating teaching and learning in tandum is entirely consistent with the requirement of solving the basic problem faced by constructivists: the reconciliation of the individual learner, the classroom (including whatever teaching is occurring), and the larger society. Thus, we will treat constructivist teaching and learning together in this section.

Arcavi and Schoenfeld [23] have taken the position that students must be viewed as quite capable of making sense of the challenges with which they are confronted. Furthermore, Confrey [24] suggests that students' methods and ideas have the status of "genuine knowledge", and can be characterized as subjective, diverse, and rational in their own way. Pirie and Kieren [22] have elaborated further, expanding these ideas into four tenets of belief that teachers must have to create a constructivist learning environment:

> 1. Although a teacher may have the intention to move students towards particular mathematics learning goals, she will be well aware that such progress may not be achieved by some of the students and may not be achieved as expected by others.
>
> 2. In creating an environment or providing oppurtunities for children to modify their mathematical understanding, the teacher will act upon the belief that there are different pathways to similar mathematical understanding.
>
> 3. The teacher will be aware that different people will hold different mathematical understandings.
>
> 4. The teacher will know that for any topic there are different levels of understanding, but that these are never achieved 'once and for all'.

Unfortunately, the descriptions given so far are still too general to give a clear idea of what happens in a classroom in which a constructivist environment is operating. Hence, we give two examples of actual learning situations in which a constructivist environment is present. The first is from Pirie and Kieren [22]:

> *Classroom 1* The teacher passes out sheets of paper (units) to be folded into halves, fourths, eights, and sixteenths. After the students have done this and discussed many aspects of this activity with each other, the teacher passes out "kits" containing numerous unit, half, etc, pieces. The teacher observes that many students are able to combine and compare fractions in this contexts but he nevertheless seeks to learn what different understandings students have. The teacher writes down some statements and drawings about 3/4, and then asks students to make 5 or more statements about 3/4. The responses are collected and analyzed and presumably used for the next lesson.

For an example in a somewhat less formal setting, Arcavi and Schoenfeld [23] describe a constructivist tutoring session in the Function Group lab at the University of California at Berkeley (the Function Group is a research group in mathematical cognition directed by Alan Schoenfeld):

> In this session a tutor and a student play a game called guess my rule with the aid of a computer program. The object of the game in this case, was for the student (grade 8) to guess the equation of a straight line. The student would state an input value and the tutor would provide the corresponding output value. This process was repeated until the student was able to guess the equation of the line (i.e. the rule). The computer was used to keep track of the input and output values and to verify guesses. Arcavi and Schoenfeld report in detail on exchanges between the tutor and the student in which the tutor attempts to understand the student's point of view, and make sense of the student's observations.

In both these examples, a central aspect of the instruction is studying what images or mental representations the student has, and then trying to help the student build a better representation if necessary. In any case, defining exactly what constructivist learning and teaching looks like has been an important area of research in mathematics education, and the interested reader can find further details in articles by Cobb, Yackel, and Wood [7], Williams [25], Pirie and Kieren [22], Arcavi and Schoenfeld [23] and many references therein.

So far in this section we have discussed definitions of constructivism, and presented two examples of constructivist teaching and learning in practice. We now turn our attention to criticisms of the constructivist model.

2.1.2.2.2 Criticisms of Constructivism

Lerman [26] has observed that the hypothesis of constructivism, namely that "knowledge is actively constructed by the cognizing subject, not passively received from the environment" is now widely accepted in the mathematics education community as a useful and productive hypothesis when thinking about how children learn. Thus, criticism of constructivism has generally come from radical constructivist who claim that constructivism does not go far enough and that, in fact, constructivism stops at a point that is not logically tenable. Thus, it might make sense at this stage to skip on to a discussion of radical constructivism, and reserve the following discussion until after the next section (on radical constructivism). However, we have chosen not to do this because as far as we can tell many radical constructivists in mathematics education came to their current view partly because of what they see as failings in the constructivist model. So we wish to follow the actual path taken by many scholars who subscribe to radical constructivism. Thus, the reader who is completely unfamiliar with radical constructivism might wish to skip on to section 2.1.2.3 before finishing the present section.

Essentially, the objection that radical constructivists have with constructivism is that in recognizing that the learner builds their own representations of mathematical knowledge, there is no turning back: we cannot logically continue to claim that there is an absolute body of mathematical knowledge to be learnt, and that each student will acquire the same representations of mathematics as their teacher.

Another way to understand the difference, according to Confrey [2] is that constructivism is essentially a theory of learning, whereas, radical constructivism is a theory of knowledge. The radical constructivist's criticism of constructivism is that a theory of learning is not enough to form a model or ideology of mathematics education. Ernst [1] has discussed in great detail the elements of an ideology of mathematics education. These elements include a theory of learning, a philosophy of mathematics, a theory of knowledge, a moral position, a theory of teaching mathematics, a theory of the child, etc. Thus, radical constructivists argue that if we adopt a constructivist theory of learning, and use by default, an absolutist philosophy of mathematics (as is commonly done) we have not adopted a consistent ideology of mathematics education.

Lerman [26] has made essentially the same point in the setting of philosophy of mathematics. Namely, the debate between constructivism and radical constructivism in mathematics educations is very similar to, if not the same debate as constructivist (intuitionist) and fallibilist are having about the nature of mathematics. Fallibilist argue that no matter how much we try to "reform" the absolutist theories of mathematical knowledge, because of Godel's incompleteness theorems (or just using logic directly) there is no way to know that any particular theory of mathematical knowledge is internally consistent without invoking a meta-theory of some sort. Therefore, absolute knowledge in the sense of absolutism is not possible.

Constructivists are well aware of the criticism that has been made of constructivism, and may even empathize with those criticisms, but for constructivists, completely giving up on an absolute framework results in too many unresolved problems. As Schoenfeld [21 p. 290] puts it:

> The leap into nonobjectivism, which lies at the core of *radical constructivism* is a giant one. Making it leads to a host of deep philosophical problems: If there is no "there" there, why do we appear to perceive it? By what means do we perceive it? How do we communicate? For those interested in instruction, it leads to equally deep practical questions, such as, How and what does one teach when the core of instruction, the subject matter, is no longer postulated to exist as an objective entity, and the standard notions of learning, based on internal representations of that external reality, cannot be called upon? More simply, if you are a radical constructivist, what in the world can you say about practical issues of instruction?

Thus, the debate between constructivists and radical constructivists is centered around issues that are partly a matter of intuition and experience and therefore remain largely unresolved.

2.1.2.2.3 Constructivism - Summary

We have characterized constructivism as the belief that the learner constructs their own representations of the subject under study. The implication for instruction is that representations held by the student must be nurtured and explored, with the teacher using a variety of ways to help the student come to realize their errors in representation, and to help the student bring their mental representations into line with representations held by an expert.

Criticisms of constructivism have mainly been made by radical constructivists, who argue that having rejected the representational duality between external and internal knowledge, and having accepted that learning is largely an internal process, constructivists are left with no choice but to adopt the radical constructivist position that all knowledge is internal.

In any case, having discussed constructivism and the radical constructivist critique of constructivism, it is high time that we properly discuss radical constructivism.

2.1.2.3 Radical Constructivism

As already mentioned radical constructivism had its origins in the 1960's when reform movements began in philosophy of science and in psychology. Radical constructivism goes beyond constructivism by making a full commitment to an internal theory of knowledge. We discuss the basic radical constructivist model in section 2.1.2.3.1, including connections with constructivism and two examples of instruction from a radical constructivist perspective. In section 2.1.2.3.2 we consider some criticisms of radical constructivism, and we conclude section 2.1.2.3 with a summary in section 2.1.2.3.3.

2.1.2.3.1 Radical Constructivism – The Basic Theory

A Brief History of the Development of Radical Constructivism

The groundwork for radical constructivism was begun in the 1960's as reform movements in both the philosophy of science and in psychology (perhaps reflecting the social turmoil of this period) began to challenge the traditional paradigms. We will only mention a few of the developments in psychology for comparison purposes, so the reader interested in the psychological roots should refer to Steefe and Kieren [5]. Thus, we will focus on the origins of radical constructivism that are based in the recent history of philosphy of science.

Confrey [2] has traced one of the origins of radical constructivism to the reform movement in the philosophy of science that began with the work of Karl Popper [27] in 1962. Popper claimed that scientific advances followed from falsification more often than from verification, so that the scientific method (which was the pillar of the philosophy of science at the time) can not be the whole story. Later, in the early 1970's, Kuhn [28] and Toulmin [29] argued that falsification (of hypotheses by null experimental results) alone could not explain scientific advances.

The re-examination of the sources of scientific advances led to the realization that a much larger picture was needed (than just scientific method) that included theories of individual knowledge, of methodologies, of standards of research and reporting, of proof, and of the social context of the scientific investigation. Of course, opposing views arose, but for our purposes, the next development of interest was the application of the analyses emerging from the philosophy of science to mathematics, first by Lakatos [30, 31, 32], and later by Kalmar [33], and Tymoczko [34, 35].

The emerging theories had a profound effect on the espistemology of science and mathematics. As Confrey [2, p. 2] puts it:

> All of the theories coalesced to change the view of science, and to [a] significant but lesser extent, mathematics, to make them vulnerable to systematic change, revision, debate, and rejection. All struggled to explain the twin processes of stability and change as they admitted relativism into the scientific and mathematic enterprise. And all of them challenged a simplistic view of objectivity; in each theory, the subjective, either as psychological process or a sociological process, was inexorably involved.

As one might expect, the new ideas spread to science and mathematics education, influencing discussions of classroom culture, and the epistomology of teaching and learning. To some extent the new thinking in science and mathematics merged with the reform movements in education and psychology, resulting in growing interest in constructivism. Thus, by the late 1970's and early 1980's there was widespread support for the need for new theories of learning in mathematics education. Of course, with any reform movement there are many variations, so it is not surprising that constructivism has so many different meanings.

The search for new models of learning rather naturally led to widespread interest in the work of Piaget, since as a biologist studying child development he attempted to incorporate many of the reforms in the philosophy of science into his theories. Piaget's work was concerned with the development of children's understanding of basic concepts such as number, time, and space, and also the phases of that development. The interest in Piaget by the mathematics education community was roughly paralleled in psychology by "the Piagetian studies" spearheaded by Van Eugen [36]. Piaget's work was particularly attractive since he had attempted to identify the mental structures and representations of children, which was exactly the information that theorist needed to develop new espistemological theories. As Confrey [2, p. 3] says:

> The effect of combining Piagetian work and the philosophy of science was to emphasize the importance of espistomogical issues and to challenge the assumption that children's worlds were simply inadequate or incomplete representations of adult worlds. Such contributions were necessary for the formation of a radical constructivist perspective in mathematics education.

Throughout the 1980's and 1990's work on radical constructivism has continued with most research focussed on the central problem of radical constructivism (discussed below). We wish to emphasize however, that radical constructivism and constructivism are fundamentally different movements, which have an overlap in their adherents and a convergence of their content up to a point.

Radical Constructivism

As was the case with constructivism, researchers have defined radical constructivism in varying amounts of detail. Kilpatrick [20] gives a definiton consisting of just two hypostheses, the first of which we have already seen as a reasonable definition of constructivism:

> 1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment.
>
> 2. Coming to know is an adaptive process that organizes one's experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.

Confrey [2] describes these ideas in much more detail, elaborating a defintion into a four point framework for radical constructivism. Confrey has used the work of von Glasersfeld [37] extensively to construct her framework. (Incidently the four point framework given here is not Confrey's modification of radical constructivism to which we referred in section 2.1.2 at the beginning of our discussion of the learning model.)

1. *Genetic Epistemology*: Knowledge develops over time. To understand a concept one needs to attend to both the overall construction of the mental representations, as well as the evolution of the representations.

2. *Radical Epistemology*: There are two key ideas in this point: a) Knowledge is fluid and constantly being constructed and modified. There is no fixed canon of knowledge, independent of the observer, to be discovered. b) To know something is to act on this knowledge, to reflect on the knowledge and the actions taken, and then to reflect and act on the reflections, etc, in a never ending construction.

3. *Scheme Theory*: Scheme theory in radical constructivist theory has its origins with Piaget. For Piaget a scheme is "whatever is repeated or generalizable in an action" Piaget (p.34 1970 cited in Confrey [3]). Schemes are a record of the parts of actions (on knowledge) that are repeatable or predictable. Key for radical constructivists is the observation of differences between, for example, the scheme of an individual child and the expected scheme for children of the same age.

4. *Model Building and the Construction of Others*: This point of radical constructivism refers to the idea that the knowledge of others (external to ourselves) is created in the same way that small children give life to objects and human qualitites to animals. The term 'models of others' is used to emphasize the fact that there is no priviledge status inherent in our knowledge of others. Radical constructivists argue that individuals make such constructions because they help us make predictions and give us more control.

Ernst von Glasersfeld [38] has written extensively about radical constructivism.

He has explained radical constructivism as follows:

> Radical constructivists redefine "the concept of knowledge as an *adaptive function*. In simple words, this means that the results of our cognitive efforts have the purpose of helping us to cope in the world of our experience, rather than the

traditional goal of furnishing an 'objective' representation of the world as it might 'exist' apart from us and our experience" (p.xiv)

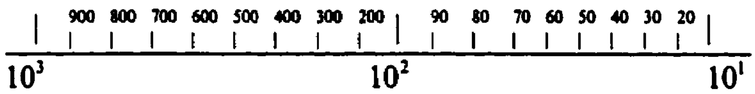
To summarize, radical constructivists agree with constructivists that knowledge is actively constructed by the learner. Radical constructivists differ from constructivists, because the former believe that there is no external reality, independent of the individual. Thus, for radical constructivists the process of coming to know is an essential part of knowing, in fact, some radical constructivist would claim that the process of coming to know and acting on knowledge is all there is to knowing, in effect, the actual facts or representations that an individual has are meaningless if considered in isolation from the process by which they were acquired or the actions that are taken on account of those representations. For radical constructivists acquiring knowledge is about acquiring control and predictability, as well as making sense of the sensory inputs and the mental reflections one has.

Having discussed the nature of radical constructivism, we now describe two classroom situations to illustrate the radical constructivist method of teaching and learning.

Radical Constructivists Models of Learning

The most difficult problem for radical constructivists is to define a practical classroom environment in which to implement the radical constructivist philosophy of teaching and learning. This problem has by no means been solved, especially the problem of how to teach from a radical constructivist perspective, so we give two examples of attempts to solve this problem, or at least some aspect of the problem.

Confrey's research group has reported work done with a college freshman, Confrey [39]. In this research the student drew a number line that was suppose to represent the entire timeline of the universe. She marked evenly spaced powers of ten on the bottom of the line and then divided each unit of a power of ten into evenly space increments from the lower power of ten to the higher power of ten on the top of the line. Thus, a portion of her timeline looked something like this:



A traditional approach would rule this incorrect since there are two different scales: The powers of ten are a multiplicative scale, and the increments of 10 or 100 are an additive scale. However, this student showed a great deal of facility calculating time intervals (with this double scale), and also showed that she could fluently switch from scientific notation (lower scale) to decimal notation (upper scale). Thus, this student showed real understanding with a non-standard model.

The challenge for the radical constructivist is where to go with this? Confrey, suggests that creating a task that would lead this student to the need to use a true logarithmic scale is the next step. Another possibility would be to devise different timelines to see where this student's methods lead.

The second example is a report by Russell and Corwin [40]. In this research teachers were asked to form conjectures about the relationship between the number of corners, edges and sides of pyramids. They were given construction materials to work with so that they could make pyramids with various shaped bases. This was all done in a classroom setting.

Russell and Corwin report that some teachers distinguished between the vertex off the plane of the base, describing this vertex as a point rather than a corner. This perspective was further reinforced by the additional exercise that teachers were asked to perform, namely the construction of two dimensional "graphs" in which all the vertices and edges were displayed. For many of the "graphs" the vertex off the plane looks very different from the base vertices.

The teachers holding the view that the vertex off the plane of the base was not a corner were encouraged to develop their idea further. Among other things it became clear that many teachers in this group thought of a corner as a point of intersection of exactly three planes (like a street corner or a room corner). The instructors of this class made no attempt to "correct" the teachers, or formally introduce the concept of 'vertex'. Instead the teachers were allowed to engage in a lively debate, thereby gaining valuable experience in the way mathematics is actually practiced.

Several more examples of radical constructivist teaching styles and classrooms are contained in Bauerfeld [41], for the interested reader. We will not discuss these examples for reasons of space limitation.

In conclusion, in this section (2.1.2.3.1) we have reviewed the origins of radical constructivism, particularly the origins in the the reform movement in philosophy of science. In addition, we looked at definitions and interpretations of radical constructivism. We noted that identification of a clear radical constructivist program is far from complete, so that we were only able to offer partial illustrations.

Having reviewed the nature of and the origins of radical constructivism, we now look at some of the criticisms that have been made of radical constructivism.

2.1.2.3.2 Radical Constructivism – Criticism

Criticisms of radical constructivism have been made on several grounds: the objectives are too extreme to be attainable, there is no grounds for explaining differences in cognition except on the basis of age, a radical constructivist model of teaching does not exist and is problematic, and finally there is a contradiction between on the one hand emphasizing the 'individual' construction of knowledge, and on the other hand requiring everyone to act "collectively" to achieve common knowledge.

Below we discuss these criticism as well as possible replies.

As we noted in section 2.1.2.2.2, Schoenfeld [21] has argued that the problems posed by radical constructivism are intractable. For example, if there is no external world, why do we appear to perceive it? How do we perceive an external world, and how do we communicate?

In our view, these questions are based on a misunderstanding of radical constructivism. Radical constructivism postulates that the individual constructs their reality to help them build their knowledge, in much the same way as a child attributes all sorts of properties to animals and objects. When the chilhood images are no longer useful, the individual adopts incresingly complex perceptions. Thus, by radical constructivism we perceive an external world because doing so helps us organize and understand subjects which we decide to study. Of course, we can strengthen Schoenfeld's objection by asking why do we do all this? What is so important about the process of organizing and understanding knowledge (which afterall, is entirely internally constructed)? As far as we can tell, radical constructivists have no answer to this question. Nevertheless, we cannot very well dismiss radical constructivism on these grounds, since no theory of knowledge (or in the case of radical constructivism, the process of acquiring knowledge) can explain why seeking knowledge is important, since this is a meta-knowledge question.

Confrey [2] has argued that radical constructivism is essentialist except with respect to age. As she puts it:

> Constructivists approaches can be criticized as positing a universalist or essentialist view of cognition across classifications except age.

Confrey argues that constructivist have not yet accounted for the differences in performance that exist between cultures, races, and genders. Furthermore, the heavy emphasis on the individual can lead to a devaluation of individuals who hold back, who work for consensus, or who for whatever reason choose not to actively participate.

One answer to this criticism is simply that individuals from different backgrounds are seeking different knowledge, so their process of organizing that knowledge is different. (Of course, it is more accurate to argue that individuals need to perceive variations in performance between people from different cultures, races, and genders, because this helps us organize our knowledge. In other words, building differences of affect between the people in our mental images helps us organize our knowledge.)

The third criticism of radical constructivism, noted by Confrey [2], and Schoenfeld [21], is that, as of this writing, there is no model of radical constructivist teaching. This is a serious problem for constructivists, because sooner or later the end product of teaching or research winds up with the 'correct' answer being imposed or at least strongly suggested. This is evident in both the examples we gave in section 2.1.2.3.1. For example, in the scale example one suggested course of action is to direct the student to a log scale, in effect, the 'correct' answer. In the other example of "discovering Euler's equation for pyramids", although this was not how the research was reported, discovering Euler's equation is in fact the task, so that anybody teaching this group would not be satisfied if Euler's equation was not eventually discovered (at least in some subsequent lesson). One would have to be a very disciplined and committed radical constructivist not to feel this way.

In defence of radical constructivism, unless we can prove that there is no teaching model in radical constructivism, the absence of a teaching model cannot be used to argue that radical constructivism is an unsound theory. The problem is pragmatic: teaching from a radical constructivist perspective is not *currently* possible, but that does not mean it is impossible.

Zevenbergen [42], is a recent critique of constructivism based on what Zevenbergen views as the contradiction between emphasizing the individual (reality is a mental construction), and postulating that we act "collectively" to build the *same* perceptions of an external reality. Zevenbergen argues, among other things, that radical constructivism can be used to justify individualism, and also to justify ignoring social injustice in schools and education in general. The argument is that since an individual constructs their internal reality to help them with the process of knowing, and since part of that construction is the perception that we all have the same images of reality, why is there any need to affect social change? Does oppression exist? What does oppression mean if everything has been constructed in the mind of the individual to aid in the construction of knowledge and understanding?

This seems like a difficult conundrum for radical constructivists, but actually, there is a 'cheap' way out: oppression (in the usual sense) is built into our mental images to help us organize our knowledge. We have images of a struggle over oppression because it helps us organize our knowledge and understanding. We take action on the images of oppression (or not) because the process of acquiring understanding is what it means to know, and taking action is part of the process. Thus, taking action to change social injustice is completely consistent with radical constructivism.

In summary, we have discussed several criticisms that have been made of radical constructivism, as well as given possible replies that illustrate radical constructivist thinking. The criticisms we studied include: the theory is intractable, there is no theory of teaching, there is no explanation for differences in affect, other than age, and radical constructivism can be used to justify maintaining the status quo. We believe all of these criticisms (except the absence of a teaching model) can be addressed by carefully applying the postulates of radical constructivism, in particular, the ideas of taking action on knowledge and the key point that the process of organizing knowledge is what is meant by understanding.

In any case, after briefly summarizing section 2.1.2.3, we will discuss Confrey's modifications of radical constructivism.

2.1.2.3.3 Radical Constructivism – Summary

We have reviewed the origins of radical constructivism from the reform movements in philosophy of science and psychology. We observed that constructivism had a different origin in mathematics education from radical constructivism, so they have developed somewhat in opposition.

We have given several definitions of radical constructivism and reconciled them into some key points such as: knowledge is actively constructed in the learner's head, taking action on knowledge is an essential aspect of coming to understand, and the process of learning is what it means to know.

The primary unsolved problem with radical constructivism is that there is no teaching model. Other criticisms seem to have reasonable answers within the framework of radical constructivism.

Having reviewed the representations model, constructivism and radical constructivism, we are now prepared to discuss the model of knowledge that we attempted to use for our research project.

2.1.2.4 Confrey's Modification of Radical Constructivism

Confrey's [2, 3, 4] modification has seven tenets. These are briefly described below:

1. *Human development depends on the environment.* In Confrey's perspective one educates at all levels for a global society that includes all living things. The whole point of constructivism is that any exclusion of part of human experience is limiting to learning and teaching. Thus, this tenet is included in Kilpatrick's [20] two assumptions discussed in section 2.1.2.3.1.

2. *The self is both autonomous and communal.* This assumption is designed to answer the well known criticism of radical constructivism of how different individuals create the same reality. Confrey is suggesting that part of our desire for knowledge is to create coherence in addition to the usually stated motive of active control and manipulation.

3. *Diversity and dissent are anticipated.* Since one of the basic natures postulated (see #2) is that humans are autonomous, we expect that there will be different opinions and ideas of how to learn and solve problems. We expect different ideas about what is relevant in any given situation. Diversity is therefore encouraged as an essential part of developing the autonomous side of human beings.

4. *Emotional intelligence is acknowledged.* By emotional intelligence Confrey means the ability to monitor one's own and other's feelings, to discriminate between them, and use the information to help one's thinking and actions. This assumption is included partly to permit the inclusion of various issues of mathematical affect as part of mathematics.

5. *Abstraction is reconceptualized and placed in a dialectic.* This postulate is intended to resolve the long running historical debate between the value of abstraction and practical knowledge. Confrey believes these two to be vital aspects of the same thing which she refers to as a dialectic.

6. *Learning is viewed as a reciprocal activity*. This statement follows from the emphasis placed on diversity. Teachers must be willing to learn new methods as they arise in the classroom. Unexpected solutions should be the inspiration of a new or modified teaching strategy.

7. *Classrooms are studied as interactions among interactions*. Essentially, the intention of this assumption is to recognize the interplay between individual and social aspects of classrooms.

2.1.2.5 Reification and APOS

In addition to the above model, we have attempted to incorporate two specific processes of learning mathematics: reification and APOS.

For a discussion of reification, see Sfard [43], Sfard and Linchevski [44], and Kieran [45] (Kieran discusses the same ideas but uses different terminology). Reification is essentially the learning process of moving from a *process* understanding of a concept to an *object* understanding of the concept. For an example of reification, consider the gradual shift of comprehension required to understand a function first as a formula for calculating an answer given an input value (process notion), then as a mapping, and finally as say, an element of a vector space (object notion).

The other learning process is APOS, which is an expansion of the idea of interiorization due to Piaget. The Research in Undergraduate Mathematics Education Community (RUMEC) has been especially active in the investigation of APOS. See, for example, Asiala, *et al* [46] or Breidenbach *et al* [47] .

APOS stands for Action Process Object Schema. In the version of APOS used by the RUMEC group there are four stages: actions are formed by manipulating previously constructed objects (mental or physical), processess are formed by *interiorizing* actions, processes are *encapsulated* to form an object understanding, and finally actions, processes and objects can be *organized* into schemas. In addition, objects can be de-encapsulated back to the processes from which they were formed. (Schema can be *thematized* into objects.) In our work we have concentrated on the first three stages, in effect, actions, processes and objects, because we regarded the process of identifying schemas to be too complex for this stage of our research.

For an example of APOS framework, again using functions, a student who is unable to even think about functions unless they are given a specific function to calculate at a specific point has an action understanding. If the student is able to think about functions as having a domain and range they have begun to acquire a process understanding. If the student learns to think about whole classes of functions, such as continuous functions, then they have encapsulated their process understanding to an object understanding.

It is not hard to see that encapsulation and reification are closely related. This has been noted by Sfard and Linchevski, [44]. Reification and encapsulation both describe the learning process of coming to understand a *process* as an *object*. Nevertheless, we do need to exercise caution when comparing reification and encapsulation: although, the idea of an object understanding of a concept is similar in both frameworks, what we mean by *process* is different in the reification format and the APOS format. In the APOS framework process is *intended* action, in effect action that has been interiorized. Since there is no concept of action in reification, process in reification includes the APOS notion of action. Thus, the set of mathematical activities that we would identify as process within an APOS format is a subset of the set of activities that we would identify as process in the reification format.

In chapter 4 we have used the Sfard-Linchevski-Kieran reification format as the basis for our analysis of what indicators of understanding we expected to see in the data. We have use the Sfard-Linchevski-Kieran format as opposed to APOS in chapter 4, because we believe that distinguishing between action and process understanding in APOS is largely dependent on the individual: we must carefully scrutinize the thoughts of the student to determine if the activity is intended (process) or being performed routinely (action). In our analyses in chapter 6, however, where we used data indicating student thoughts, we have used the APOS format when there was enough data to warrant attempting to identify the interiorization of actions into processes.

To summarize, we regard reification and encapsulation as essentially the same thing. We have used the Sfard-Linchevski-Kieran reification format to analyze what is meant by understanding in chapter 4 and chapter 6, but we have sometimes used APOS in chapter 6 when identification of the process of interiorization was possible.

2.1.3 Model of Mathematical Knowledge

To complete our background model for this research project we needed a model of mathematical knowledge. We have attempted to use the feminist point of view collected into a model by Burton [48] (see also Hanna, [49]). Burton drew heavily from the work of feminist scholars working in the area of philosophy of science. Five aspects of mathematics knowledge were identified by Burton:

1. Its person and cultural/social relatedness.
2. The aesthetics of mathematical thinking it invokes.
3. Its nurturing of intuition and insight.
4. Its recognition and celebration of different approaches particularly in styles of thinking.
5. The globality of its applications.

The challenge of applying Burton's model was substantial, and we were not too successful. Our main attempt consisted of asking ethonographic questions, not to learn the demographics of the students we interviewed, but because according to the Burton model ethonographic information is an essential part of understanding and assessing the mathematical knowledge of the students we interview (point 1 above). Unfortunately, we had no guidelines about how to use the information collected or even what questions to ask. Since we did not solve this problem, we chose to omit the ethnographic data from this thesis.

On the other hand, during interviews we tried very hard to be sensitive to and encouraging of different approaches to solving the problems we posed (point 4 above).

Having discussed the model of mathematical knowledge that we attempted to use in this study, we next consider what we expected the results to be.

2.2 Expectations before Entering the Study

In this section we will summarize what we expected the results of this study to be, prior to the study. As already mentioned, we expected the classes to be organized as lectures with an informal atmosphere. We anticipated that by third year university, most students would be convinced that a proof is the only way to know that a theorem is correct. We expected that few if any of the students would have given much thought to what it means to know mathematics. We thought that few if any students would have enough awareness of their own culture to be able to analyze how their culture was affecting what they were learning, how they were learning it, and what they understood to be mathematical knowledge.

Awareness of gender issues has improved in the last few decades (Leder, [50]), so we expected that a few students might have at least wondered how their gender was affecting their experience in the classes studied. We expected to observe practically everyone struggle with the process of reification, at each stage in the course. Finally, we had some general ideas about where students would have difficulties, but we had no detailed picture of the problems students would have.

Since a central focus of this research has been to ascertain what sort of mental images students have of the subject area, we expected a fair amount of diversity in understanding and problem solving approach.

2.3 Summary

In this chapter we have described the theoretical framework that we used for this study. We discussed the teaching model that we expected to find used in the classes studied, a learning model, and a model of mathematical knowledge. In addition, we have listed what we expected some of the results of this study to be, before we began the study.

Chapter 3

Literature Review

3.1 Research in Mathematics Education on Complex Analysis

We have reviewed what research results have been published in the area of teaching and learning complex analysis. We have found none that were explicitly related to our study. CARL and ERIC searches, as well as searches of the SFU library's connection to British data bases, turned up no results. Keyword searches did turn up 15-20 papers, but they all were concerned with difficult statistical problems in educational research.

We have found a few references on the subject of complex numbers. Tirosh and Almog [51] studied 78 high school students in Israel. The students had 8 lessons (the length of the lessons is not specified in this paper) on complex numbers, including an explanation of why the usual ordering < on the real numbers does not hold for the complex numbers. 94% of the students passed a summation examination on complex numbers at the end of the 8 lessons.

Tirosh and Almog found that the students they studied had a very difficult time understanding that complex numbers were numbers (many students insisted that a number has to represent a quantity). In addition, they had great difficulty understanding that the usual ordering relation on the real numbers does not hold for the complex numbers. For example, in a post-test administered by Tirosh and Almog 95% of the students agreed that $i < 4 + i$. Students explained that when a positive number is added to a number it makes the number larger, so $i < 4 + i$.

We have not pursued research reports on high school students actively, since we have assumed (with reasonable justification we believe, based on our data) that third year university students have a different set of difficulties when learning complex analysis. For example, even though we did not specifically study the question of whether or not the complex numbers are numbers, we found no evidence during interviews or tutoring sessions that the students studied did not recognize complex numbers as numbers.

The students in our study did have problems understanding that the ordering relation, $<$, does not extend to the complex numbers, but most of the difficulties we found had to do with what it means to extend $<$ (the students we studied had no instruction on ordering at the time of the interview). So, although there is some superficial similarity between the responses of high school students and the students in our study, careful scrutiny shows distinctly different problems of understanding. For example, several of the students in our study were side tracked by the modulus ordering (discussed in section 4.2.5.4) and almost all the students in our study rejected any order on the complex numbers that extends $<$ once they understood what it means to extend the ordering.

Thus, all that we were able to do is to model this study after similar studies in other areas of university level mathematics. We were primarily interested in research methodology and frameworks for analyzing the data, when we reviewed other research reports.

3.2 Related Research

Some projects from which we have used ideas for this research are Zazkis [52], Dubinsky, Dautermann, Leron, and Zazkis [53], Williams [54], White and Mitchelmore [55], Thompson [56], Confrey and Smith [57], Breidenbach, Dubinsky, Hawks, and Nichols [47], Zazkis and Campbell [58], and Zazkis and Dubinsky [59]. We have also used some general ideas from Schoenfeld [60] and Tall [61]. Finally, we have used Hillel and Sierpinska [62, 63] to help us understand the contrast between complex numbers and linear algebra in R^2, in effect, deepening our understanding of representations in complex analysis.

Zazkis [52] studied the mathematical behaviour of college students faced with the challenge of finding the inverse of a compound element. A compound element is, for example, a product of two invertible objects. Thus, if A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$. Zazkis posed the questions in a Logo environment. We used this study as one of our models for our methodology, since we planned to use clinical interviews. We have also used this paper as a model of reporting and analyzing our results.

Dubinsky, *et al*, [53] reports on student knowledge of group theory. The subjects in this study were 24 high school teachers taking a summer refresher course in abstract algebra. We used this paper to get ideas for the format of interview questions, and get a sense of what difficulty of questions yields useful results. The analysis in this paper was also instructive for us, as a model of how to take data and identify process thinking, object thinking, etc.

Williams [54] identified various models of limit held by college students (the exact level is not specified in this paper) at a large American university. The Williams study had two phases: a short written questionnaire given to 341 students, and second stage detailed study of 10 students selected from the first stage. We have used various adaptations of the questionnaire used by Williams. The format consists of three parts. Part A consists of six true/false statements, part B asks subjects to decide which of the statements in part A is most accurate, and part C asks students to describe in their own words the concept being studied in part A. We explored subjects' understanding of seven topics using the question format used by Williams (see appendix 4 for the exact questions).

White and Mitchelmore [55] studied changes in student skill level in doing related rate problems before and after 24 hours of concept based instruction in calculus. We did not ask any questions about related rates in our study, but we used this study to get a sense of how complicated questions could be and still yield interesting data. For example, one question subjects were asked to answer in the White and Mitchelmore study involved a vehicle attempting to get from point A to point C by traversing a stretch of open country followed by a stretch of highway: the road is longer, but faster, so some combination of terrain is the fastest route. Finding the fastest route is the objective of the problem. This was evidently a difficult problem (only 4 attempts out of 40 were successful), but the results revealed the limitations of the instruction given to prepare the subjects.

The White and Mitchelmore study also gave us a perspective of what range of difficulty was useful: this study had a specific goal of comparing subjects' skill level on four different versions of each of four questions. The versions consisted of a word problem with differing amounts of translation into symbols done for the subjects in the study. We followed this example in our study, asking both very open questions of discovery, such as "Why are poles isolated?" to basic manipulation questions, such as "Find the real part of $4 + i$".

Thompson [56], studied the understanding that 19 senior and graduate mathematics students (most were in teaching programs) had with the fundamental theorem of calculus. Thompson found that limited understanding of rate of change and function covariance was the main difficulty students had understanding the fundamental theorem. This is a very detailed paper, requiring an intense effort to absorb: we used this paper as justification for attacking difficult concepts (of which there are many in complex analysis). That is, Thompson [56] is a paper that reports on research done with mature subjects on a complicated aspect of understanding an important theorem of calculus. Thus, our attempt to study equally complicated problems was justified by previous research. In addition, Thompson [56] has several interview excerpts which we studied as part of our effort to learn how to do qualitative interviews.

Confrey and Smith [57] is primarily about rates of change. They argue that rates of change is a more natural approach to the function concept than the set correspondence notion of a function that is usually given to calculus students. However, Confrey and Smith also comment on a number of implications of radical constructivism that they believe have not been properly incorporated into many research efforts. They believe that in many cases student concepts are actually correct or at least internally consistent, even though they are different from the usual formulation of the concept under consideration. For example, they cite the example of the term "constant": some students may think of a "constant function" as a function with a constant rate of change.

Since radical constructivists recognize that individuals construct their knowledge, Confrey and Smith believe that we should be exploring the nature of student conceptions. They argue that we should assume these conceptions are correct (if perhaps unusual), rather than investigating them in terms of how they are mistaken, or in terms of how to help students avoid "misconceptions". Furthermore, they disagree that improved teaching necessarily means that more students will develop the established mental images of the concept under consideration.

We have tried very hard in our study to give the subjects what Confrey calls "voice" and to be aware of the context of the interview, what Confrey calls "perspective". Perspective is primarily all of the knowledge and expectation that the researcher brings to the interview. Unfortunately, we were not able to use Confrey's interview framework as much we would have liked, because we found that most of our subjects were very concerned with developing an established understanding of the material covered in class.

Breidenbach, Dubinsky, Hawks, and Nichols [47] reports on research done to study how students develop a process notion of functions in a learning environment that includes computers with the programming language, ISETL. This paper was our introduction to the idea of encapsulation. We also used this paper to get a sense of what kinds of questions to ask in interviews, since there are several interview excerpts included in this paper.

Zazkis and Campbell [58] is a study of the understanding held by preservice teachers of the divisibility and multiplicative structure of the natural numbers. We closely followed the data collection procedures described in this paper. Furthermore, the many interview excerpts included in this paper, gave us additional insight into how to conduct our interviews.

In addition, Zazkis and Campbell [58] was very useful to us because the APOS framework (explained in section 2.1.2.5) was used to begin construction of a complete genetic decomposition of their participants' understanding of divisibility and the multiplicative structure of the natural numbers. We found it very helpful to us to see how to look at the data and extract evidence of action, process, or object understandings of divisibility.

Zazkis and Dubinsky [59] is principally a discussion of research done on subjects from an abstract algebra class. This paper is specifically focused on the problem students had interpreting successive permutations of the dihedral group, however, other examples of the same problem of interpretation are also discussed. We used this paper for its examples of interviewing technique (interview excerpts), and as an example of how to analyze our data.

Schoenfeld [60] is an extensive article on problem solving. We were particularly impressed by the time-line graphs for problem solvers of various skill levels. We used these diagrams in a general way to help us estimate how mathematically sophisticated the students were that we interviewed. For examples of two of these diagrams see figure 3.1.

The difference in time use between an expert and an experienced problem solver is remarkable. Although we did not record time use measurements we attempted to notice how much time (relatively) students spent on the problem solving activities listed in figure 3.1.

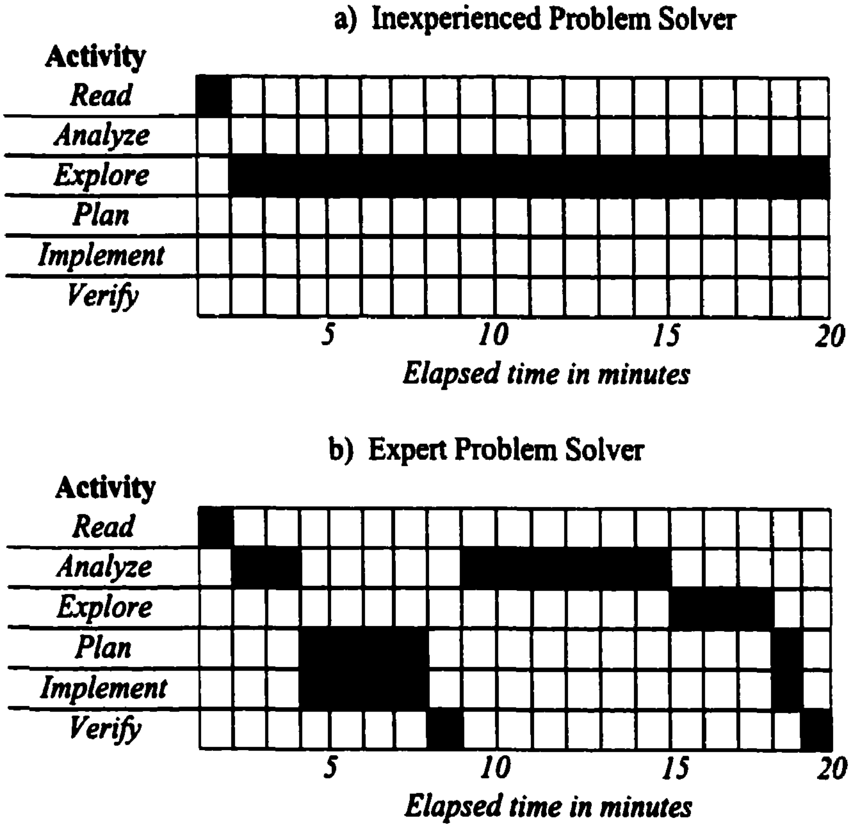


Figure 3.1
Time use diagrams for a) a student problem solver and
b) an expert problem solver working on a difficult problem.
Adapted from Schoenfeld [60].

Tall [61] is a review of recent research (mostly as reported in American journals, up to 1991) on functions, limits, infinity and proof. We used this paper to get some idea of what sort of responses we could expect from students if we asked them questions about these topics. For example, Tall confirms (citing several other authors) the work of Williams [54] that many student difficulties with the concept of limit stem from the conflict between everyday usage of the word limit and the technical meaning of limit in mathematics. He argues that the definition of a limit, while precise mathematically, is not a good *cognitive root*: the definition of a limit is not built on concepts that students are familiar with and which they can readily extended to further mathematical development. When interviewing students we looked for signs of what might be a good *cognitive root* for the student on questions of limit, continuity, analyticity , et cetera, in effect, on all the questions modeled after the Williams questionnarie (see above or see appendix 4).

At the conclusion of our research we found it very useful to contrast work in linear algebra on the representations of a linear operator with respect to different bases with our work on representations of complex numbers. Hillel and Sierpinska [62, 63] studied the problem of representation of a linear mapping expressed as a matrix, $L:R^n \rightarrow R^m$. In general, vectors are expressed in terms of two different bases, *e* and *f* (the basis vectors are of course, expressed in terms of the standard bases for R^n and R^m, respectively). Hillel and Sierpinska analyzed student responses to questions that involve calculating the value of a linear mapping acting on basis vectors as well as more general vectors. The students worked under several sets of circumstances. They found that although practically all students correctly realized that $L(e_i)$ involved the i^{th} column of the matrix for L, many forgot that they needed to express their answer in terms of the basis for R^m. So $L(e_i)_k = \sum_{j=1}^{m} L_{ji} f_k^j$ where $L(e_i)_k$ is the k[th] component of the vector $L(e_i)$, L_{ji} is the element in the j[th] row and i[th] column in the matrix of L, and f_k^j is the k[th] component of the j[th] basis vector for R^m. More specifically, one cannot just read-off the answer as the i[th] column of L. A central aspect of this type of problem, identified by Hillel and Sierpinska is the confusion students experience between the images of the basis vectors and the *representations* of these images in the basis *f*.

In contrast, the representations of complex numbers that we have studied (the representations are studied in great detail in chapter 4, section 4.2) are essentially representations of the addition and multiplication operations (we realized this after studying Hillel and Sierpinska's paper). Although the polar form can be seen as a coordinate transformation of the Cartesian form, because different variables are used, coordinate changes when needed are apparent: we observed no confusion amongst students about which "coordinate" system they were using. Furthermore, it is practically never useful (at least in a beginning course) to make a change of coordinates within a given representation. For example, we almost never consider a transformations such as $u = \frac{2x+iy}{2}$, $v = \frac{x-2iy}{2}$, because this sort of coordinate system makes it harder to find the real and imaginary parts of a complex number such as (1, 2) in the (u, v) system. (The one major exception to this is complex conjugate coordinates, which are very useful. However, complex conjugate coordinates are a coordinate system in the tangent space of each point in the complex plane, so this is quite a different situation.)

The similarity between complex number representations and linear algebra is with *bilinear* forms, since the multiplication and addition operations for the complex numbers are bilinear forms. This is a difficult subject that we will not discuss further.

3.3 Conclusion

We have not found any reports in the mathematics education literature on complex analysis, with the exception of a few papers on complex numbers. Accordingly, we modeled our research after research done in other areas of university level mathematics, such as algebra and calculus.

Chapter 4

History and Epistemology of the Complex Numbers

4.1 Introduction

In this chapter we examine the historical development of complex numbers, with particular attention focused on the development of the various representations. We have organized this chapter into a section on epistemology and a section on history, since we wish to emphasize the time sequence of the historical development. This organization makes sense because we will see that, historically, there were really only two major conceptual obstacles in the development of complex numbers: that they made any sense at all (they were "impossible" or "imaginary") and the search for a geometric interpretation. Evidently, as early as the 17^{th} century, all the major mathematicians had no difficulty performing calculations with complex numbers, using whatever techniques were available to them. In other words, questions such as, what is the meaning of e^{it} were not seriously debated (at least, as far as we can tell from the record of the history of mathematics).

Nevertheless, we have included a section on the epistemology of complex numbers for completeness, and because, irrespective of whether or not research mathematicians had any difficulty understanding the basic form and meaning of complex numbers, students at third year level certainly do have difficulty with these concepts.

In our discussion of the epistemology of complex numbers we have included both a modern treatment of the mechanics of each of the four representations discussed, as well some general comments on what is meant by understanding these concepts. Our analysis of what understanding means is restricted to a discussion of what reification looks like in the present context. (Reification is discussed in chapter 2.)

4.2 Epistemology of the Complex Numbers and their Representations

In this section we discuss the meaning of the various representations of complex numbers. We have included discussions of the algebraic extension form, $z = x + iy$, the vector form, $z = (x, y)$, the polar form, $z = re^{i\theta}$, and the symbolic form, in which complex numbers are simply represented by z. In addition, we include discussions of some general questions such as ordering, direction and the modulus, which are relevant to all the representations under consideration.

4.2.1 The Algebraic Extension Representation: $z = x + iy$

In this section we discuss the algebraic extension representation of complex numbers. We have divided this section into four parts: basics, meaning and difficulties, understanding, and a summary of this section. In the subsection on basics we discuss how the algebraic extension representation works and the context of this representation with respect to the real numbers. The subsection on meaning addresses questions such as, "What does the + sign mean in $z = x + iy$?".

In the subsection on understanding we have attempted to identify indicators of reification (recall that reification is the primary aspect of understanding that we are studying in this thesis). This section is necessarily speculative since the actual evidence of reification, if any, is in the data collected. Thus, an analysis of the data which suggests reification has taken place, is contained in Chapter 6.

4.2.1.1 The Algebraic Extension Representation - Basics

The algebraic extension representation of complex numbers emphasizes the fact that the complex numbers are an algebraic field extension of the real field. The notation is borrowed from the development of the field of algebraic numbers as an extension of the rationals. Thus, $Q(\sqrt{2})$ consist of all of the numbers needed to make the set $Q \cup \{\sqrt{2}\}$ a field. These numbers can be expressed in the form a + b$\sqrt{2}$, where addition and multiplication indicated in this expression are defined in terms of addition and multiplication of the rationals in such a way that all of the usual field laws hold. In particular, the distributive law is retained: $(a + b\sqrt{2})(c + d\sqrt{2}) =$
$ac + 2bd + (ad + bc)\sqrt{2}$.

Of course, to understand what it means to multiply a rational number times $\sqrt{2}$ we need to invoke one of the developments of the real numbers, such as Dedekind cuts: $\sqrt{2}$ is defined as the set of rational numbers less than $\sqrt{2}$, and if b is a positive rational number, b$\sqrt{2}$ is the set of all rational numbers formed by multiplying elements of $\sqrt{2}$ by b. If b $\leq$ 0, multiplication is more complicated, see Rudin [64] for details.

To see that each nonzero number of the form a + b$\sqrt{2}$ has an inverse, we can find an explicit formula: the inverse of a + b$\sqrt{2}$ is $\frac{a - b\sqrt{2}}{a^2 - 2b^2}$. Note that for a and b rational, the denominator of this expression cannot be zero (a and b cannot both be zero if a + b$\sqrt{2}$ is nonzero). Thus, Q($\sqrt{2}$) is a field.

Further construction of the real number field (including the transcendental numbers) would take us too far away from our main topic, so interested readers should consult a text on real analysis, such as, Rudin [64]. In any case, given the real numbers we can construct the complex numbers as an algebraic field extension of the real numbers, in the sense that i is not a real number, but is the root of the polynomial $x^2 + 1$.

We proceed just as in the case of algebraic extensions of the rationals: every complex number is represented as a number of the form a + ib, where a and b are real numbers. The multiplication and addition in this expression are extensions of the usual real number operations, defined in such a way as to preserve the usual field laws of the real numbers:

$(a + ib)(c + id) = (ac - bd) + i(ad + bc)$.

4.2.1.2 The Algebraic Extension Representation - Meaning

We can give a geometric representation of complex addition and multiplication (discussed below), but the existence of plausible (and useful) geometric representations does not change the fact that complex addition and multiplication are far removed from the basic interpretations that we have for addition and multiplication when working with whole numbers. Thus, we cannot expect to have the same sort of picture of an expression such as, ib that we have for 3 times 2. There is nothing new in this situation, since e times π also has no easy interpretation. The point is that we must abandon the expectation of having clear interpretations of complex addition and multiplication, at least in the sense of having interpretations that relate directly to everyday life in the way that addition and multiplication of whole numbers do (for example, 3 times 2 is the total number of apples in 3 bags that contain 2 apples each). Nevertheless, abandonment of this sort of everyday interpretation of multiplication by i did not come easily for the mathematics community, and the whole question of the meaning of i and multiplication by i is a large part of the discussion in section 4.3, below.

Nonetheless, this is not to say that we can assign no meaning to an expression such as bi. The meaning is simply abstract, in effect, bi means b units of i. This says very little about bi, but it is the relationship between bi and other numbers that gives the real meaning (recall from sections 2.1.2.3 and 2.1.2.4 that relationships were a key part of knowing). For example, $ib \times ib = -b^2$, so that ib is a square root of $-b^2$. The main conceptual difficulty in this statement is overcoming the belief that negative numbers do not have square roots. Of course, this belief is only true for the real numbers, so the difficulty can be expressed as the problem of fully accepting or understanding that we are working in a larger system of numbers than the real numbers.

The advantages of the algebraic form are the ease in performing basic calculations (especially addition), the ease in learning about complex numbers, and the clear emphasis of the relationship between the reals and the complex numbers, in the sense that complex number computations are done in terms of the usual real number operations. In addition, the algebraic form is generally the best form to use for solving basic algebraic problems, such as factoring a polynomial, or finding roots of a specific polynomial (an exception occurs in the case of roots of unity [$z^n = 1$] for which the polar form is very efficient). For example, to solve the polynomial equation $z^2 + c_1 z + c_2 = 0$, c_1 and c_2 complex numbers, we would have to use the quadratic formula, which, in general, will give us an answer that is a complex number in the algebraic form.

There is a pedagogical aspect of the algebraic form that was present in all the classes studied and the textbooks used for these classes: the algebraic form was combined with the geometric interpretation of the complex numbers in a plane and sometimes used as a vector form. Although it was not stated explicitly, implicitly 1 was sometimes used as though it was the unit vector **i** in $\Re^2$ and i used for the unit vector **j**. This is not much of a problem at a beginning level, but at a more advanced level can lead to problems understanding that the representation of the complex numbers as a plane has limitations. We will return to the question of treating the algebraic form as a vector form in much more detail later (in Chapter 6).

4.2.1.3 The Algebraic Extension Representation – Understanding

In this section we attempt to anticipate what the results of our research will be by analyzing some of the possible steps in the reification process for the algebraic extension representation. In other words, we wish to identify indications of *process* and *object* understandings of this representation that we think we will see in the responses that students give to our questions. In addition, we would like to answer the question, "What indicators show that shifting from one understanding to the other (reification) is occurring?" As we mentioned in section 2.1.2.5, in chapter 4 we have restricted our discussion to Sfard-Linchevski-Kieran model of reification (discussed in section 2.1.2.5) since we believe that distinguishing between *action* and *process* understandings in the APOS model requires a careful analysis of the data.

Certainly, indicators of a process understanding would be the ability to do calculations (particularly, without skipping steps), recognition that x and y are real in the expression $z = x + iy$, and recognition that the multiplication rule,

$(a + ib)(c + id) = (ac - bd) + (ad + bd)i$, in this representation is just the usual real number distributive law, with the additional feature that $i^2 = -1$.

The problem of identifying indicators of an object understanding is more difficult. Certainly the ability to shift from the algebraic representation to any of the other representations would be one indication, but are there indications *within* the algebraic extension representation? In other words, can we find clear indications of an object understanding if we ask questions that only involve the algebraic extension representation for their solution? We think the answer is no, since having an object understanding of a concept partly means having the ability to consider the concept amongst other similar concepts as objects. Thus, to show an object understanding of the algebraic extension representation necessarily means showing the ability to think about this representation within the context of other representations of complex numbers, or possibly in the context of other field extension (algebraic or otherwise) of the real numbers. Since the latter possibility was not investigated in this study, we are left with "thinks about the algebraic representation within the context of other representations of complex numbers" as our main group of indicators of an object understanding (by "think" we mean builds mental images and relationships in the sense discussed in chapter 2). This is a group of indicators, since we expect to find student thinking about the different representations in a wide variety of ways. For example, we expect to see most students show fluency with the algebraic extension representation, good skill with the polar representation, and various levels of judgment of when to shift between two representations. Another possibility is that a student thinks geometrically, so that they show very little process skill with the algebraic extension representation but use the polar and symbolic representations fluently. Thus, we should expect to see a variety of combinations of representations used by students who have an object understanding of two or more representations.

Since some of the students interviewed may not achieve a full object understanding of the algebraic extension representation it is worth considering whether there is some way to think of the algebraic extension representation as a collection of objects that might be individually understood as objects. We think the answer is yes to some extent. The algebraic extension representation is composed of objects such as the real numbers, i, the operations of complex multiplication and addition, and various algorithms, such as the algorithms for division or calculating inverses. Thus, a student might be able to understand any of these objects, as objects, without demonstrating more than a process understanding of the whole representation.

In addition, to the various indicators of either a process or object understanding we also expect to see indications of reification occurring. For example, we believe that a student who can streamline the algorithms, for example, for calculating the inverse of $z = x + iy$, is showing signs of reification, because streamlining an algorithm shows that the student notices they are performing a process that can be made more efficient, so they are thinking of the whole process, which means they have begun to acquire an object understanding of the process. Thus, a student who streamlines an algorithm understands that they are using an algorithm, and that like other algorithms they may know, it can be modified if necessary. This is clearly showing an object understanding of the algorithm.

To summarize, using the Sfard-Linchevski-Kieren model, we expect that a *process* understanding of the algebraic extension representation is indicated by preoccupation with the details of basic calculations. An *object* understanding of the algebraic extension representation is indicated by fluency with computational details combined with at least some judgment of when to shift from the algebraic extension representation to some other representation. Of course, these are general considerations, and whether or not any particular student has a process or object understanding requires close study.

4.2.1.4 The Algebraic Extension Representation - Summary

In summary, the algebraic form $z = x + iy$ emphasizes the complex numbers as an algebraic field extension of the real numbers. The advantages of this representation are that it is relatively easy to learn and apply (since it does not require much preparation other than algebra). The main disadvantage of the algebraic form is that the very simplicity makes complicated calculations, involving large exponents, very tedious. In addition, the multivalueness of the complex numbers is not easily represented in the algebraic representation, so that finding roots is not easy.

The main conceptual difficulties with the algebraic form is understanding what ib means and understanding that the new complex operations of + and × have been defined in terms of the + and × operations for real numbers in a way that retains all the usual rules. The surprising ease with which we can extend the usual rules for real numbers sometimes masks the fact that we are working in a new system of numbers, and some errors can be traced to thinking in real number terms as opposed to thinking in complex number terms.

Finally, we expect to identify a process understanding of the algebraic extension representation as preoccupation with the details of computations, whereas an object understanding will reveal a fluency with computations and an awareness of when to shift to another representation. We have also identified some indications of reification.

4.2.2 The Cartesian Vector Representation: z = (x, y)

We now turn our attention to the Cartesian vector form. We have broken the discussion into the same four parts that we used in section 4.2.1: basics, meaning, understanding and a summary.

4.2.2.1 The Cartesian Vector Representation - Basics

In the Cartesian vector representation the complex numbers are treated as a vector space of vectors in $\Re^2$ over $\Re$. Vector addition is defined as follows: Given $z = (x, y)$ and $w = (u, v)$, with x, y, u, and v all real, then $z + w = (x + u, y + v)$. The addition inside the parentheses is the usual real number addition. Scalar multiplication is defined by $r(x, y) = (rx, ry)$. In this equation, the multiplication inside the parentheses is the usual real number multiplication. Thus far we have only $\Re^2$ treated as a vector space over $\Re$.

To get the complex numbers we need to introduce a vector product, defined as follows (with the above definitions for z and w): $zw = (x, y)(u, v) = (xu - yv, xv + yu)$, where the addition and multiplication operations inside the parentheses are the usual real number operations. Then it is possible to show that the vector product, together with vector addition defined above forms a field structure on $\Re^2$. For example, the multiplicative identity element in this field is (1, 0), since

$(1, 0)(x, y) = (1x - 0y, 0x + 1y) = (x, y)$. The inverse of any element (x, y), not equal to zero is $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right)$. Note that for real x and y, with $(x, y) \neq (0, 0)$ the denominator in this expression cannot be zero.

We can also see that there is an element in this field with the property that $z^2 = (-1, 0)$, namely (0, 1): $(0, 1)(0,1) = (0\times0 - 1\times1, 0\times1 + 0\times1) = (-1, 0)$. The significance of this fact becomes clear when we understand that the real numbers are represented by the set of numbers of the form (a, 0), so that if $z^2 = (-1, 0)$, then z is the square root of -1. That the real numbers are represented by numbers of the form (a, 0) follows readily from the observations that $(a, 0) + (b, 0) = (a + b, 0)$ and $(a, 0)(b, 0) = (ab - 0, 0) = (ab, 0)$, so that the mapping from $\Re$ into the complex numbers given by $a \to (a, 0)$ is a field isomorphism.

Since the Cartesian vector form is somewhat more abstract than the algebraic extension form it is important to note that the two forms are isomorphic by way of the mapping $(x, y) \to x + iy$. For example, under this mapping, $(x, y)(u, v) = (xu - yv, xv + yu) \to xu - yv + (xv + yu)i = (x + iy)(u + iv)$, so the multiplication operation is preserved. A similar calculation shows that the addition operation is preserved (clearly $(x, y) \to x + iy$ is one to one and onto).

It is very important to recognize that the algebraic structure of the complex numbers is built into the vector form z = (x, y) (by way of the vector product). Thus, the real number y and (0, y) are not the same thing (unless y = 0), since (0, y) includes the implied structure of the complex numbers, whereas, the real number y has no such implied conventions attached to it.

In fact, the real number y is represented as (y, 0) in the vector form, and (y, 0) is definitely not equal to (0, y), unless y = 0.

4.2.2.2 The Cartesian Vector Representation – Meaning

As in the case of the algebraic extension form, there is a geometric interpretation of the Cartesian vector form available. For reasons that will be discussed in section 4.3, finding a geometric interpretation of the complex numbers was deemed to be an important problem by many mathematicians of the 17^{th}, 18^{th}, and 19^{th} centuries, but the Cartesian vector form was introduced (by Hamilton) specifically to avoid geometry. This point will be discussed further in section 4.3.

Nevertheless, as already noted above, the existence of geometric or algebraic representations of the complex numbers, while helpful, does not by itself give meaning to the Cartesian form. As was the case for the algebraic extension form, we can also build understanding from the relationships between the various objects in the vector space we are using to represent the complex numbers. For example, to understand complex multiplication using the Cartesian vector representation we need to study the vector product, (x, y)(u, v), asking such questions as: is $|zw| = |z||w|$?, how does the angle between z and w affect zw?, does zw = wz?, is f(z, w) = zw continuous?, and is f(z, w) = zw bilinear?. These are the kinds of questions one would ask if studying a vector space.

However, in our study of the Cartesian vector form we asked questions at a more basic level: in particular, we focused on the fact that the symbol i does not appear in the vector form. We also asked students questions about the fact that the scalars in the Cartesian representation are real numbers, and that the square root of −1 is represented by (0, 1), so we do not use i in this representation.

In any case, to properly study the complex numbers as a vector space of $\Re^2$ over $\Re$, we need to study vector spaces, linear transformations, and the techniques of linear algebra for understanding vector spaces. A full discussion of these topics would take us too far afield.

The main advantage of the Cartesian vector form is that the complex numbers are described as a vector space. This enables us to use all the tools of linear algebra to understand the complex numbers. The emphasis in this representation is on the complex numbers as a formal system. Unfortunately, at third year level most students are not mathematically sophisticated enough to appreciate how to use the techniques of linear algebra to help them understand complex numbers.

Thus, the formal nature of the Cartesian representation is also its main disadvantage. For example, to multiply zw = (x, y)(u, v) one has to know the multiplication rule, but to multiply (x + iy)(u + iv), we can use the rules for real numbers, together with $i^2 = -1$. In the latter case, one can think of the + in x + iy as ordinary addition, without serious consequences. Hence, (x + iy)(u + iv) looks very similar to (a + b)(c + d), for real numbers a, b, c, d, so that the algebraic representation is easily related to previous knowledge.

4.2.2.3 The Cartesian Vector Representation - Understanding

As was the case in section 4.2.1.3, we will discuss understanding within a narrow framework of reification. Aside from the general indicators of a *process* understanding identified in section 4.2.1.3 (in effect, preoccupation with computational difficulties), the Cartesian vector representation has an additional process level difficulty: namely, the formalism of the ordered pair notation. Thus, a process understanding is indicated not only by computational difficulties, but also by not using the formulism correctly, or at least not using the formalism fluently.

For example, mistakes indicating that a student was struggling to obtain a process understanding would includes things such as treating (0, y) as though it was real, translating to the algebraic form incorrectly, or bringing i into the calculation. A specific example of this kind of mistake would be something like this: (3, 5)(0, 2) = 2i(3, 5) = (6i, 10i). Although we could make sense of this expression, in terms of the Cartesian vector form (6i, 10i) is not correct because there is no symbol i in the Cartesian vector form.

We expect the indicators of an *object* understanding of the Cartesian vector form to be essentially the same as the indicators already identified for an object understanding of the algebraic extension form in section 4.2.1.3: fluent computational skill, clear understanding of the formalism, and good judgment about when to shift to another representation. Of course, other indications of an object understanding would include consideration of the complex numbers as a vector space, and the use of general properties of vector spaces to make conclusions about the complex numbers, but we did not expect to see these indicators in a beginning complex analysis class.

Thus, the only new indicator that we need to look for with the Cartesian vector representation is skill with the formalism.

4.2.2.4 The Cartesian Vector Representation – Summary

In summary, the Cartesian vector representation emphasizes the complex numbers as a vector space of vectors in $\Re^2$ over $\Re$ with a vector product defined as $(x, y)(u, v) = (xu - yv, xv + yu)$. The main advantage of this representation is that any reference to roots of negative numbers is avoided. The complex numbers can be understood as a formal system in this representation. Thus, any conceptual problems that students have accepting roots of negative numbers can be avoided, at least, in the beginning. There is no need to understand i, since i is not used in the Cartesian vector form.

The formal approach is also the main disadvantage of the Cartesian vector form that is particular to this form: students may find the formal approach hard to understand and think about. In addition, the Cartesian vector form suffers from the same problems as any rectilinear form: complicated products are much easier to do in the polar representation, and the multivaluedness of the complex numbers is hard to represent in the Cartesian representation.

The primary additional indicator of understanding that we have with the Cartesian vector form is the skill level with the formalism. In short, does the student understand that we do not have i in the Cartesian vector representation.

4.2.3 The Polar Representation of Complex Numbers: $z = re^{i\theta}$

The polar representation is a vector form in polar coordinates. The main advantages of the polar form are ease of calculating products of complex numbers, perfect representation of the inherent multivaluedness of the complex numbers, and the ease of simple geometric applications in polar coordinates. We have divided this section into the usual four parts, basics, meaning, understanding and a summary.

4.2.3.1 The Polar Representation of Complex Numbers - Basics

In the polar form, $z = re^{i\theta}$, r is the modulus of z, i.e., for z = (a, b), $r = \sqrt{a^2 + b^2}$.

e is the base of the natural logarithm, defined as $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. The expression $e^{i\theta}$ is defined formally as follows: $e^{i\theta} = \cos\theta + i\sin\theta$. This expresses $e^{i\theta}$ in terms that we already understand, since sin θ and cos θ are real. In other words, we formally define $e^{i\theta}$ in terms of the algebraic extension representation, with x and y expressed in terms of polar coordinates. The modern derivation of this formula, known as Euler's formula, comes from the theory of power series in the complex plane. A proper development of the theory of complex power series is beyond the scope of this section, but interested readers can consult Saff and Snider [65], or for a more advanced treatment, see Ahlfors [66] or Nehari [67]. In section 4.3 we discuss Euler's formula in more detail.

At first sight it might seem more logical to define $z = r(\cos\theta + i\sin\theta)$ to avoid the use of complex exponents, which we have not introduced at this stage. Perhaps even more consistent would be to use the vector representation in polar coordinates, $z = r(\cos\theta, \sin\theta)$. This last definition would clearly define the polar representation as a vector representation in polar coordinates. However, computations are easier in the exponential formalism, than in either of the trigonometric formalisms suggested above, so we will follow the development of the polar representation used by all of the textbooks for the classes studied Priestley [68], Churchill and Brown [69], and Saff and Snider [65], namely $z = re^{i\theta}$.

Once we accept the exponential formalism, it is important to realize that Euler's equation is much more than simply a formal expression: we can use the rules of exponents in the usual way to simplify calculations. For example, we can derive the rule for multiplying two complex numbers in the polar representation: if $z = re^{i\theta}$ and $w = ve^{i\varphi}$, then

$zw = rve^{i\theta}e^{i\varphi} = rve^{i(\theta+\varphi)}$. Thus, the rule for multiplying two numbers in the polar representation is: add the arguments to get the argument of the product, and the product of the moduli is the modulus of the product of the two complex numbers. This simple rule makes multiplication in the polar representation especially convenient. We have more to say about Euler's equation in sections 4.2.3.2 and 4.3.

Unfortunately, there is no simple rule for adding in the polar representation. The only thing that can be done is to convert into Cartesian vector form, add, then convert back to the polar form. Thus, given that $z = re^{i\theta}$ and $w = ve^{i\varphi}$, to find z + w, we express z as (r cos θ, r sin θ) and w = (v cos φ, v sin φ), then z + w = (r cos θ + v cos φ, r sin θ + v sin φ). This can be converted to polar form

$z + w = ue^{i\beta}$, where $u = \sqrt{(r\cos\vartheta + v\cos\phi)^2 + (r\sin\vartheta + v\sin\phi)^2}$, and

$\beta = \arctan\dfrac{r\sin\vartheta + v\sin\phi}{r\cos\vartheta + v\cos\phi}$. To choose the correct value of the arctan (x) function we examine the signs of $r\cos\vartheta + v\cos\phi$ and $r\sin\vartheta + v\sin\phi$.

The polar form is derived from the algebraic form, x + iy, in the sense that i is given (as the $\sqrt{-1}$) and appears explicitly in the polar form. We also have infinitely many non-trivial representations of i in the polar form, namely, $i = e^{i(\frac{\pi}{2}+2\pi k)}$, where k is any integer. At first sight it seems odd (or even incorrect) that we have a representation for i that involves i, but we must remember that we can always refer back to a vector representation in polar form, so that

$$i = \left(\cos\left(\frac{\pi}{2} + 2\pi k\right), \sin\left(\frac{\pi}{2} + 2\pi k\right)\right) = \left(0, \sin\left(\frac{\pi}{2} + 2\pi k\right)\right) = (0, 1) \ .$$

4.2.3.2 The Polar Representation of Complex Numbers – Meaning

The most difficult part of the polar form is understanding what we mean by e to the power of iθ. Of course, it is not necessary to develop a theory of complex exponents to understand $e^{i\theta}$ because we are considering a special case: the case of a real base and a pure imaginary exponent (the polar angle θ is always real). In this special case, we can avoid the difficulties of complex exponents, by defining $e^{i\theta}$ in terms of Euler's equation: $e^{i\theta} = \cos\theta + i\sin\theta$. Then the rules for manipulating exponents can be derived using the angle addition and subtraction rules for sine and cosine. For example,

$$e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = (\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi + \sin\phi\cos\theta)$$
$$= (\cos(\theta+\phi) + i\sin(\theta+\phi) = e^{i(\theta+\phi)} \ .$$

The other important rule for working with exponents, can also be derived using the properties of sine and cosine: $\left(e^{i\theta}\right)^n = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta) = e^{in\theta}$. The middle equation, $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$ is known as DeMoivre's equation and can be proved using mathematical induction.

For our purposes we need look no further, because we have interpreted the polar formalism in terms that we have described in sections 4.2.1 and 4.2.2. Nevertheless, it is important to note that the polar representation turns out to be the best representation to use to formulate the inherent multivaluedness of the complex numbers. This suggests that Euler's equation is much deeper than the treatment we have given it. We will return to this point in section 4.3.

4.2.3.3 The Polar Representation of Complex Numbers – Understanding

In this section, as in previous sections (4.2.1.3 and 4.2.2.3), we wish to identify indicators of *process* and *object* understanding (in the Sfard-Kieren model of reification discussed in chapter 2), as well as, indications of *reification*.

As was the case with the algebraic extension and vector representations, the main indicator of a process understanding should be a preoccupation with computational details. In this case we have a complication, namely trigonometric related difficulties, that was not present in the representations previously discussed. Recall that in the algebraic extension and Cartesian vector representations, complex multiplication and addition were defined in terms of real number multiplication and addition. Since we expect students at third year university to be completely fluent with arithmetic (or perhaps use of their calculator), and reasonably fluent in algebra, any difficulties they experience with the algebraic extension and Cartesian vector representations, at a process level of understanding, should be due to the representation and not lack of preparation.

We must be careful with the polar representation however, because many students even at third year university level may not be conversant in trigonometry. Trigonometry, especially the arctan(x) function is an important part of the polar representation, so a student who is poorly prepared in trigonometry is bound to have trouble using the polar form. Thus, when looking for indications that a student is "preoccupied with computational" details we should be circumspect about difficulties that appear to be trigonometric in nature.

Another important feature of calculations in the polar representation that is not particularly evident in the other representations is work with exponents. Even at third year level we expect some students to have trouble with exponents.

Thus, some of the computational difficulties that we expect to see, other than trigonometric and exponent difficulties, are: using r^2 instead of r (it is common for students calculate the modulus as r^2, forgetting to take the square root), difficulty finding the argument, θ (at least expressed in terms of trigonometric functions), choosing the sign of θ correctly, and attempting to add two numbers directly in the polar representation.

In short, the difficulty we expect to have analyzing indicators of a process understanding is that student work will be confounded by lack of preparedness with trigonometry and exponents. Accordingly, our best approach is probably to look for indicators of reification and an object understanding first, and if these are not present, decide if the student has at least acquired some aspects of a process understanding.

Aside from the ability to shift from one representation to another, indications of *reification* and an *object* understanding of the polar representation are: a clear understanding that we cannot add directly with the polar representation, combining the polar representation with geometric arguments, and the ability to work with $e^{i\theta}$ as opposed to cos θ and sin θ. For an example of using the polar representation with a geometric argument, consider the task of computing z^{-1}, if $z = re^{i\theta}$: One way to answer would be to say $z^{-1} = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{e^{-i\theta}}{r}$. However, we believe considerably more understanding is revealed by an answer such as, "to find z^{-1} reflect z radially about the unit circle, so that the new modulus is $\frac{1}{r}$ and multiply by the unit conjugate, to get $z^{-1} = \frac{1}{r}e^{-i\theta}$". This is not to say that a student who uses the first method does not have an object understanding. The first method simply does not show anything more than a process understanding. The second method shows innovation, since this method is not normally taught in class, hence a student who used such an argument would be showing that they can think about the polar representation in conjunction with geometric thinking.

Incidentally, geometric thinking, although always present as a possible tool, is much more prominent in the polar representation than in the algebraic extension or Cartesian forms, partly because of the need to use pictures to understand the trigonometric relationships, but also because the type of problems that we use the polar form to solve commonly have a strong geometric component. In other words, the reason we choose the polar form to do a problem is because we have noticed some sort of radial symmetry that can be exploited using the polar form.

In conclusion, we expect a *process* understanding to be indicated, as usual by a preoccupation with computational details, but we must be aware that computational difficulties in the polar representation may be due to inadequate preparation in trigonometry and/or exponents. An *object* understanding of the polar representation is indicated by the ability to shift representations effectively and the ability to incorporate geometric thinking. Some signs that we identified that reification is taking place are: understanding to switch to Cartesian vector or algebraic extension form to add and doing most calculations with $e^{i\theta}$.

4.2.3.4 The Polar Representation of Complex Numbers – Summary

The polar representation of complex numbers, $z = e^{i\theta}$, is a vector representation in polar coordinates. In addition, we use Euler's equation, $e^{i\theta} = (\cos\theta + i\sin\theta)$, essentially as a convenient computational aid. We noted that there is no need to develop a general theory of complex exponents to understand $e^{i\theta}$, since we are using a very special case. The rules for calculating products and integral powers are derived from the corresponding rules for sine and cosine.

As with the other representations studied so far, we expect that the primary indication of a *process* understanding will be a preoccupation with computational details. However, we have noted that we expect this indicator to be confounded, in some cases, by difficulties with trigonometry and use of exponents.

We expect an *object* understanding to be indicated by fluent use of the polar representation, good decisions about when to shift, and possibly incorporation of geometric arguments into the solution of problems.

4.2.4 The Symbolic Representation of Complex Numbers

The symbolic representation of the complex numbers emphasizes the overall structure of and general relationships between the complex numbers. In the symbolic representation complex numbers are simply represented as z. We have divided this section into the same four parts that we divided previous sections: basics, meanings, understanding, and a summary.

4.2.4.1 The Symbolic Representation of Complex Numbers - Basics

As already mentioned, in the symbolic representation, complex numbers are simply denoted by z. Two operations are introduced, namely + and ×, which give the complex numbers a field structure that has all the algebraic properties of the real numbers.

In the symbolic representation we are allowed to freely multiply z by any complex number in the algebraic extension form. For example, an expression such as $z^2 + (1 - i)$ is properly expressed in the symbolic form. Thus, the symbolic form is somewhat more advanced than the previous forms, since we are taking for granted that we understand, for example, the expression (1 – i), in effect, a number in the algebraic extension form. In the symbolic form the complex conjugate and complex conjugation are used frequently: $\bar{z}$ is the reflection of z across the real axis, so that $\operatorname{Re} z = \frac{\bar{z} + z}{2}$ and $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$.

We can establish various facts in the symbolic representation by using any of the other representations. For example, the modulus of z, denoted by $|z|$, is given by

$|z| = +\sqrt{z\bar{z}}$, and $z^{-1} = \frac{\bar{z}}{|z|^2}$ if z is not zero. Thus, if $z = re^{i\theta}$, $\bar{z} = re^{-i\theta}$, then

$\sqrt{z\bar{z}} = \sqrt{(re^{i\theta})(re^{-i\theta})} = \sqrt{r^2} = r$.

The symbolic form is especially useful when combined with geometric thinking. For example, to solve the equation $|z - i| = |z + 1|$, we observe that if z is a solution to this equation then z is equidistant between i and –1. Thus, Re z = - Im z. . To solve this equation using, for example, the algebraic form we have to solve the equation: $x^2 + (y - 1)^2 = (x + 1)^2 + y^2$. Then y = -x is the only solution.

4.2.4.2 The Symbolic Representation of Complex Numbers – Meaning

The symbolic representation does not require any new explanations of meaning since all operations can be understood in terms of one of the other representations. However, the full utility of the symbolic representation lies in not directly relating z back to one of the other representations. In other words, once the basic rules of the representation have been established, the symbolic representation is most useful in those situations where we want to express relationships between complex numbers that are independent of the properties of the real sub-field. Hence, to be fluent with the symbolic representation a student needs to be able to think "z", and only think "z = x + iy" to verify basic results or perhaps to find the exact answer to a specific problem.

Thus, we need to address the question of what it means to think "z", in effect, think of complex numbers abstracted from the real sub-field. Unfortunately, it is difficult to answer this question in terms of basic concepts that students can reasonably be expected to grasp in a beginning course on complex analysis. For example, to think "z" we need to have some understanding of the one point compactification topology of the extended complex plane, experience with fields, and to have made some progress towards separating our thinking of complex numbers away from just R^2. Put another way, thinking in terms of z requires us to begin expanding our picture of the complex numbers beyond "R^2 with some additional structure such as vector multiplication". Since this is quite sophisticated we will not discuss the meaning of thinking in terms of z further since it is very unlikely that we will actually see much evidence of any symbolic thinking, let alone thinking at the level of sophistication being described here.

Thus, we will restrict our attention to evidence of a *process* understanding in the next section, since we do not expect to see a full *object* understanding of the symbolic form.

4.2.4.3 The Symbolic Representation of Complex Numbers – Understanding

As already mentioned, we expect that some students may be able to acquire skill manipulating expressions in z. Hence, we might see a *process* understanding, manifested by the use of symbolic rules for simplification, as opposed to using one of the other representations for computations. For example, a student who has a solid process understanding of the symbolic form would recognize that an expression such as $\left|\frac{\bar{z}-i}{z+i}\right|$ is just one, since the numerator and denominator are complex conjugates. Thus, a student who simplifies this expression by substituting $z = x + iy$ has not shown a process understanding of the symbolic form.

The main tools of symbolic manipulation are the modulus or distance and conjugation. Since these concepts are present in the other representations, we do not expect to see a student who attempts to use the symbolic form struggle with the modulus or conjugates. Rather, different levels of skill with the symbolic form are more a matter of breadth of the method, as opposed to degree of preoccupation with computational details. So for example, a student might be able to use the notion of distance represented by the modulus to solve a problem symbolically (without much trouble), but not recognize conjugation when it occurs in an expression. Thus, we need to look for evidence of fluency with several techniques such as, use of the modulus, conjugation and application of geometry to determine the level of process understanding that a student has.

4.2.4.4 The Symbolic Representation of Complex Numbers - Summary

In the symbolic representation of complex numbers we treat complex numbers as z. The connection between the complex numbers and the real numbers plays a much less prominent role than in the other representations we have discussed.

Since the level of experience with complex analysis needed to make the meaning of the symbolic representation completely clear is far beyond a beginning course, we have restricted our discussion to noting that the basic ideas needed for the a process understanding of the symbolic form can all be understood in terms of the other representations. These basic concepts include the modulus, conjugation and basic geometrical ideas, such as the equations of lines and circles in symbolic form.

We do not expect to see more than a *process* understanding of the symbolic form. Since a student who is attempting to use the symbolic representation has most likely already mastered one or more of the other representations, we do not expect to see students struggle with symbolic computations. Instead, we expect different skill levels to be revealed as the number of basic operations (modulus, conjugation, applications of geometry) that the student can use.

4.2.5 Geometry, Order and Direction in the Complex Plane

In this section we discuss three additional aspects of the complex numbers that are not directly associated with any one of the four representations discussed, namely geometry, order and direction in the complex plane.

By geometry we mean relationships between complex numbers that can be determined by using the properties of R^2. Since a beginning course does not cover the extended complex plane thoroughly enough for students to use the geometry of the extended complex plane, we will confine our discussion to the geometry of R^2.

There is no ordering on the complex plane, at least there is no ordering that is an extension of the ordering on the real numbers, but in this section we discuss what this means. Since there is no ordering on the complex plane, we cannot define direction either. This is much harder to see, so we will discuss this point at some length.

We have organized this section into three subsections, one subsection for each of the three topics of this section. However we have not gone into as much detail as in previous sections, because we have not collected enough data on the topics of this section to warrant such a thorough analysis. Thus, the focus in the discussion below is on basics and meaning.

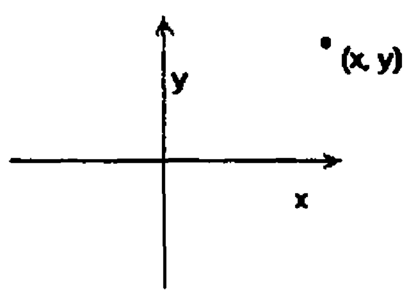
4.2.5.1 The Geometry of the Complex Plane

In this section we briefly review the geometry of the complex plane. Although the topology of the complex plane (particularly the extended complex plane) is very important for a thorough analysis of analytic and meromorphic functions, in a beginning course there is simply not enough time to cover topological developments in any detail, so we will not cover the topological perspective.

By R^2 we mean the set of ordered pairs, (x, y), where x and y are any real numbers. We can represent this set as all points in a plane. We introduce a Cartesian coordinate system, by drawing two straight lines that intersect in a right angle, in the plane of R^2. The intersection point is called the origin and has coordinates (0, 0). We choose one of the lines and call this line the x axis. We denote the direction of the x axis that points in the direction of the right half of the plane as the positive direction. The other line is denoted the y axis, with positive direction pointing to the left of the x axis (see Figure 4.1). Then each point in R^2, (x, y), can be located in the plane by measuring x units from the origin, in the positive direction if x is positive, in the negative direction if x is negative, and then moving y units from the x axis in the positive or negative direction, depending on whether y is positive or negative respectively.

Figure 4.1

The Cartesian Coordinate Axis with a typical point labeled.
The arrows indicate the positive directions



A very important aspect of Cartesian coordinates in R^2 is that the point (x, 0) is located a distance |x| from the origin, so that the notion of (Euclidean) distance on a *line* is built into the Cartesian coordinates system.

So far, we have described R^2. To get the complex plane we have to introduce addition and multiplication operations that have been described in sections 3.2.1, 3.2.2 and 3.2.3. We can add additional structure to the complex plane by defining a distance function: If $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ are points in the complex plane, then d: C×C → R defined by $d(z_1,z_2)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, is a distance function on C. Then if one of the points is zero, this formula gives the modulus of the remaining point. One of the many wonders of complex analysis is that $d(z_1, z_2)$ is completely compatible with the complex multiplication operation in the sense that the modulus of a product is equal to the product of the moduli. This is quite surprising, since the complex multiplication operation is a complex bilinear form on C ($\times$:C $\times$ C $\rightarrow$ C), whereas $d(z_1, z_2)$ has none of the nice properties characteristic of bilinear forms. (An important example of a bilinear form on R^2 that is not compatible with $d(z_1, z_2)$ in the sense discussed here, is the usual dot product: $|\mathbf{v} \cdot \mathbf{u}| \neq |\mathbf{v}| \cdot |\mathbf{u}|$.) Further discussion of this point would take us too far a field, since we want to consider the modulus in connection with ordering and direction. Hence we will next discuss ordering.

4.2.5.2 There is No Order (that extends <) in the Complex Plane

Although it is possible to order the complex plane topologically with, for example, a lexicographic ordering, by ordering the complex numbers we mean with an ordering that extends the usual "less than" (<) ordering on the real numbers. Thus, if such an ordering exists it must not only satisfy set theoretical requirements of any linear ordering (triangle inequality and trichotomy), but also satisfy the usual algebraic properties of < on the real numbers. Briefly, these rules are:

1. If x and y are positive, then $0 < xy$.
2. If x is positive and y is negative $xy < 0$
3. If $x < y$, then $(z + x) < (z + y)$.

From these three rules we can derive the more familiar forms: If $x < y$, and $0 < c$, then $0 < y - x$, by rule 3, so $0 < c(y - x)$, by rule 1, hence $cx < cy$. On the other hand, if $c < 0$, then by rule 2 we have $c(y - x) < 0$, so $cy < cx$, yielding the usual rule of reversing the direction of the inequality when multiplying by a negative number. Using these rules we can readily show that there is no extension of < to the complex numbers: Suppose there is such an extension. By trichotomy, $i = 0$, $i < 0$, or $0 < i$. Clearly $i \neq 0$, since $i^2 = -1$, but $0^2 = 0$. If $i < 0$, then multiplying both sides of this inequality should reverse the order, so $0 < i^2$, or $0 < -1$. Since this last statement is false, we cannot have $i < 0$. On the other hand, if $0 < i$, then multiplying by i preserves the order, so $0 < i^2$, $0 < -1$. This is also false, so we are left with no possible way to satisfy both the operational rules given above and the requirements of a linear order (trichotomy). Thus, we are not able to order the complex plane.

4.2.5.3 There is No Direction in the Complex Plane

Although the absence of an ordering in the complex plane that extends < of the real numbers is readily shown, it is much harder to recognize that the complex plane also has no meaningful notion of direction. This is one of the shortcomings of the representation, previously discussed, of the complex numbers as points in the plane. For example, the real and imaginary axes appear to define directions that are clearly orthogonal. The problems with ordering, and direction of, the complex numbers are much easier to understand if we use a sphere (stereographic sphere) to model the complex numbers (a punctured sphere if we do not include the point at infinity).

In any case, using the complex plane, it appears that any line passing through the origin defines a direction. To see that this view is untenable in the whole plane, recall that the rules for manipulating transfinite numbers (in an unordered vector space) are: 1. $k \pm \infty = \infty$, 2. $\infty \pm \infty = \infty$, 3. $k \cdot \infty = \infty$, and 4. $\infty \cdot \infty = \infty$.

Then if we represent a line through the origin by the equation z = (x, kx), where k is a real number, then $\lim_{x \to \infty} z = (\infty, k\infty) = (\infty, \infty)$, using rule 3, so that there is only one point at infinity, which implies that in the geometry of the complex plane all lines through the origin have the same direction. We note in passing that since this is a matter of topology, there is no direction in R^2 either. However, note that R^2 with the lexicographic order does have some directions. For example, (-∞, -∞) < (-∞, ∞) < (∞, -∞) < (∞, ∞) are different points in the extended lexicographic plane.

Of course, we can still define direction locally using the coordinate axes, but we do not have direction in any neighbourhood that includes the point at infinity, in effect, for whole (half) lines.

4.2.5.4 The Modulus

The modulus has been considered in connection with each of the representations discussed in previous sections of this chapter, however we would like to discuss the modulus in connection with ordering and direction. It is possible to define equivalence classes on the complex plane by using the modulus: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, then we define $z_1 \sim z_2$ provided that $r_1 = r_2$. Using the polar representation it is easy to see that we have defined an equivalence relation. (Recall that ~ is an equivalence relation on a set S, if for any x, y, z ∈ S, we have x ~ x; if x ~ y, then y ~ x; and if x ~ y and y ~ z, then x ~ z.) We can then define operations on the equivalence classes by $[z_1][z_2] = [z]$, where [z] is the equivalence class for which $r = r_1 \cdot r_2$, and $[z_1] + [z_2] = [z]$, where [z] is the equivalence class for which $r = r_1 + r_2$. With these definitions the set of equivalence classes is isomorphic to the non-negative real numbers, so that we can say $[z_1] < [z_2]$ provided $r_1 < r_2$.

This last point is the point we want to make, because we found that several students in our study had a very robust belief that this ordering on the equivalence classes "of circles centered at the origin" (C, ~) defines an ordering on C. We will have more to say about this in sections 6.4.3 and 6.4.4. Suffice it to say, for the purposes of this section that the ordering defined on (C, ~) definitely does not define a linear order on the complex numbers (no distinction is made, by this ordering, between different complex numbers on the same circle).

4.2.5.5 Geometry – Summary

In section 4.2.5 we have discussed the epistemology of some aspects of the geometry of the complex plane. In particular, we have discussed the Cartesian coordinate system and the absence of an ordering on the complex plane, that extends the usual ordering on the real numbers. We have also discussed the absence of a notion of direction on the complex plane (except locally). Finally, we briefly examined the set of equivalence classes on the complex plane defined by "circles centered at the origin" and considered the order induced on this set by the order on the non-negative real numbers.

We also noted that the topics in this section (except for Cartesian coordinates) are on the "edges" of a standard third year course, since the extended complex plane is not a topic of study.

Having concluded our brief study of the epistemology of the geometry of the complex plane, we are ready to summarize the epistemology of the complex numbers and their representations.

4.2.6 Summary of the Epistemology of the Complex Numbers

We have discussed at great length the epistemology of four representations of complex numbers, namely: the algebraic extension representation, $z = x + iy$; the Cartesian vector representation, $z = (x, y)$; the polar vector representation $z = re^{i\theta}$; and the symbolic representation in which complex numbers are represented by simply z.

In each case we discussed the basics of the representation, the meanings of the various aspects of each representation, and speculated on what it means to understand a representation with a *process* and an *object* understanding within the learning model described as reification. As well, for each representation we attempted to identify indicators of the process of reification.

Finally, we considered some aspects of the geometry of the complex plane that overlap with all four representations. These are the Cartesian coordinates system, the absence of ordering and direction on the complex plane, and the set of equivalence classes of concentric circles about the origin.

Having thoroughly discussed the aspects of the epistemology of the complex numbers that are particularly relevant to our thesis, we are now ready to study the history of our present day understanding of complex numbers.

4.3 The History of the Complex Numbers

We have divided this section into 3 periods, namely the period up to about 1650, the period from 1650 to about 1800, and the nineteenth century. The first period was an age of conflict and curiosity: on the one hand practically all the leading mathematician were sure that complex numbers were imaginary or impossible, but still, there seemed to be something there. From 1650 to 1800 was a period of discovery of the utility of complex numbers. The nineteenth century was a period in which the complex numbers were finally understood. The overall period we will cover is from antiquity to about 1880.

All three of these periods are characterized to varying degrees by three themes:

1. Epistemological questions - Do complex numbers exist? If so, what do they mean and how should we interpret them? If not why do they seem to "work" in some cases?

2. The search for a geometric model of the complex numbers. For the overall period of this account of the history of complex numbers, a clear geometric model would have been accepted as proof of the existence of complex numbers by most mathematicians. To what extent proof of the existence of complex numbers, as opposed to the desire for an aid to conceptualization was the motivation for the search of a geometric model is unclear from the historical record.

3. The search for an abstract, or algebraic formulation of the complex numbers. The emphasis of this program was to avoid any reference to $\sqrt{-1}$, and thereby side step deeply held beliefs that no such number exists.

The first period was dominated by theme 1. The second period was dominated by theme 2, although to a somewhat lesser extent because of the vast number of applications of the complex numbers that were invented in this period. The third theme was of interest in the third period, although it was overshadowed by the first two themes. Thus, themes 1 and 2 were far more significant than theme 3. On hindsight, we can see that the hope that introduction of an abstract algebraic formulation of complex numbers would act as an "end run" around the prejudice that the complex numbers were "impossible", was overly optimistic!

An important sub-theme that was prevalent in period 2, but was also evident in the work of Cauchy (period 3), was the discovery of applications of complex numbers. This theme dominated most of period 2, and was characterized by an attitude that can be expressed (somewhat whimsically) as: "we don't know what they mean, but they sure are useful!".

In any case, we begin our account of the history of complex numbers in antiquity.

4.3.1 The First Period – Antiquity to 1650

It is not known who first observed a complex number, but Boyer [70] has noted that the observation was almost surely done in connection with using the quadratic equation. As is well known, quadratic equations, such as $x^2 + x + 1 = 0$ have no real roots. This fact manifests itself when using the quadratic formula by the appearance of a negative discriminant. Since the quadratic equation yields solutions that are clearly real or not real, we can dismiss all those equations that do not have real roots if we wish. Thus, there was no impetus to understand the imaginary or impossible roots, and indeed they were ignored for at least two millennia.

Everything changed (although, only gradually) with the discovery of formulas for solving cubic equations. Nahin [71] attributes the discovery of a formula for the depressed cubic equation, $x^3 + px = q$, $(p, q > 0)$ to Scipione del Ferro. The truly inspired idea (there is no record of how del Ferro thought of his solution, although 40 years later Cardano did give the motivation for this choice) is to let $x = u + v$, substitute this into the equation and extract two equations in u and v, by comparing the depressed cubic with the identity $(v+u)^3 - 3uv(v+u) = v^3 + u^3$. Then v can be eliminated resulting in a sixth degree equation in u that is quadratic. The result of this procedure is the real solution:

$$x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}$$

It is not too hard to show that a depressed cubic with positive coefficients in the form given above must have two complex roots. Accordingly, the complex roots could be dismissed leaving the unique real solution. Thus, del Ferro's formula did not by itself inspire any progress in the development of complex numbers.

Important progress was made however, when Nicolo Tartaglia discovered in 1535 how to solve cubic equations of the form $x^3 + px^2 = n$, $(p, n > 0)$ (Nahin, [71], Eaves [72]). Later, in 1545, Girolamo Cardano published the solution of a general cubic equation, $x^3 + a_1x^2 + a_2x + a_3 = 0$. Cardano's solution relied on the method discovered by Tartaglia, and Tartaglia claimed he was the discoverer of the general solution. In any case the idea was to let $x = y - \frac{1}{3}a_1$, then we get a depressed cubic equation with

$p = a_2 - \frac{1}{3}a_1^2$, and $q = \frac{-2}{27}a_1^3 + \frac{1}{3}a_1a_2 - a_3$. We can then use the del Ferro formula for a depressed cubic to solve this equation (by the time Cardano published his result p and q were no longer required to be positive). Gradually the solution method came to be known as the Cardano solution, so we will refer to the "Cardano solution or formula", henceforth.

Cardano recognized that certain choices of p and q lead to negative numbers inside the square roots. He freely manipulated such quantities, treating them like real numbers, but he was stumped by the problem of calculating a cube root of a complex number (Nahin, [70]). For example, he tried a solution in the form

$\sqrt[3]{2 + \sqrt{-121}} = u + \sqrt{-v}$, but this leads to two cubic equations in u and v, which have solutions that involve the cube roots of complex numbers.

The next important development was the observation in 1572 by Bombelli that some depressed cubics had real solutions even though the Cardano formula gave a solution involving complex numbers. Laugwitz [73], Boyer [70], and Nahin [71] attribute this observation to Bombelli, but Nahin claims that Cardano was also aware of this fact. Bombelli managed to derive all the rules for multiplication and addition of complex numbers, such as, (+1)(i) = +i, et cetera (Fauvel and Gray, [74]).

In any case, by the time of Cardano's death in1576 it was known that the so called irreducible case, that is the case when $\frac{q^2}{4}+\frac{p^3}{27}<0$ in the Cardano solution $x=\sqrt[3]{\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}-\sqrt[3]{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\frac{p^3}{27}}}$, always had real roots (Eaves, [72]). Bombelli considered the equation $x^3 - 15x - 4 = 0$: using the Cardano formula $x=\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$, but he also noticed by inspection that x = 4 is a root. He observed that $\left(2\pm\sqrt{-1}\right)^3=2\pm 11\sqrt{-1}$, so that $x=2+\sqrt{-1}+2-\sqrt{-1}=4$. Unfortunately, very little further progress was made in understanding complex numbers for the next century or so. However, in 1591 Viète showed that irreducible cubic equations could be solved without involving complex numbers (Nahin, [71]).

None of the sources we investigated discussed the origins of the fundamental theorem of algebra, but there is a close connection between the increased interest in the fundamental theorem of algebra and the acceptance of complex numbers. Without complex numbers the best we can do is factor every polynomial with real coefficients into factors that are at most quadratic. Thus, the mathematicians of this period were faced with a choice: accept complex numbers or give up on the fundamental theorem of algebra (the fundamental theorem of algebra says that every polynomial in C, of degree greater than zero, has a root) .

On this note we conclude our account of the first period, and move to the second period.

4.3.2 The Second Period – 1650 – 1800

The second period was a period in the history of mathematics in which the development of new mathematics and applications accelerated dramatically. This period began with work by John Wallis and Issac Barrow that built on the work of Descartes to form a geometric foundation for the development of calculus by Newton and Leibniz. Proof of the fundamental theorem of algebra became a major objective in this period.

In the 1700's many ingenious applications of complex numbers were found by people such as, Euler, Lagrange, and d'Alembert. Thus, complex numbers were used fluently in several representations even though they were thought to be impossible. Thus, as we shall see, the second period was a time of invention of many applications, but no real progress was made on the question of the status of complex numbers. In any case, it is worth looking at this period in more detail.

The period 1650 – 1800 was, of course, dominated by the invention of calculus by Newton and Leibniz. However, there was a connection between the calculus and the fundamental theorem of algebra, namely the method of partial fractions for integrating rational functions. Laugwitz [73] reports that Leibniz was interested in the fundamental theorem of algebra for this reason. Leibniz fell into the trap of assuming that polynomials with complex roots could be uniquely factored into factors of at most degree

2. He claimed that $x^4 + a^4$ had no such factorization, since no pair of the four linear factors $x \pm a\sqrt{i}$ and $x \pm ia\sqrt{i}$ is real quadratic. However, it can be seen by direct calculation that $x^4 + a^4 = \left(x^2 + a^2 + \sqrt{2}\, xa\right)\left(x^2 + a^2 - \sqrt{2}\, xa\right)$, showing that Leibniz was incorrect.

Nevertheless, according to Boyer [70], Leibniz did correctly conjecture that if f(z) is a polynomial with real coefficients, then $f(x + \sqrt{-1}\, y) + f(x - \sqrt{-1}\, y)$ is real. Leibniz was perhaps led to his conjecture by his study of equalities such as, $\sqrt{1 + \sqrt{-3}} + \sqrt{1 - \sqrt{-3}} = \sqrt{6}$. Nahin [71] observes that Leibniz thought that this equality was remarkable and that even though Leibniz was familiar with Bombelli's 1572 text on algebra, he claimed to be the first to notice this type of equality, or at least this particular equality.

We have already noted that Newton evidently took complex numbers seriously since he tried very hard to prove the fundamental theorem of algebra. Newton's good friend De Moivre is credited with De Moivre's equation, $(\cos x + i \sin x)^n = \cos nx + i \sin nx$, for the case where n is a positive integer. De Moivre apparently did not discover this equation, for example, this equation was known to Newton as early 1676 (Nahin, [71]), so it is not clear why this equation carries his name (Eaves, [72]).

The other work of note that took place in the early part of period two was the many attempts by Wallis to invent a geometric representation of complex numbers. He attempted to extend the work of Descartes. Descartes had shown how to represent the roots of a quadratic equation with real roots as points on a circle that is determined by coefficients of the quadratic. Wallis searched for an extension of this idea and published the results in 1685 (Nahin, [71]). He was not happy with his work, however, and made many other attempts, but was not successful.

In the first half of the 1700's there was less interest in a geometric interpretation of the complex number since there was generally less interest in geometric interpretations, Laugwitz [73]. The belief was that diagrams could be misleading, but also there was a general rejection of everything classical (as part of the social times, but also with the birth of calculus), and geometry was seen to be classical.

The eighteenth century was heavily influenced, if not dominated by the work of Leonard Euler, and he found many results in the basic theory of complex numbers. Euler is, of course, credited with inventing the equation (that bears his name) $e^{ix} = \cos x + i \sin x$, for x real. Boyer [70] says Euler derived this equation by adding the series for cos x and i sin x, however, Nahin [71] notes that in a letter to Bernoulli Euler explains that he found two solutions to the initial value problem

$\frac{d^2y}{dx^2} + y = 0, \quad y(0) = 2, \quad y'(0) = 0$, namely $y = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$ and $y = 2\cos x$, so he concluded that $2\cos x = e^{x\sqrt{-1}} + e^{-x\sqrt{-1}}$. By changing the initial conditions slightly we can use the same method to derive the additional result that, $2i \sin x = e^{ix} - e^{-ix}$. From these two equations we can derive Euler's equation. Nahin [71] explains that Euler did not notice the connection with power series until later (1748).

In any case, a year later in a letter to Goldbach, Euler noted that

$\frac{2^{\sqrt{-1}} + 2^{-\sqrt{-1}}}{2} \approx \frac{10}{13}$ correct to five decimal places, Nahin [71]. This shows that Euler's thinking was not confined to base e, but it is not known what led him to observe this result.

Euler studied the properties of all the basic functions of calculus when the domains were extended to complex numbers.

We give some examples:

Euler resolved the controversy raised by the computation,

$\log(1)^2 = \log(-1)^2 \Rightarrow 2\log(1) = 2\log(-1) \Rightarrow \log(1) = \log(-1) = 0$, by noting that $e^{i\pi} = -1$, so that $\log(-1) = i\pi$. Then we can see that the second equation in this sequence of steps is wrong because we have switch branches of the complex logarithm:

$$\log(1)^2 = \log(-1)^2 \Rightarrow 2\log(1) - 2\pi i = 2\log(-1) \Rightarrow 2\ln(1) + 4\pi i - 2\pi i = 2\ln|-1| + 2\pi i$$
$$\Rightarrow 0 + 2\pi i = 0 + 2\pi i \Rightarrow 2\pi i = 2\pi i$$

For this calculation we have used the branch of the logarithm with $\frac{\pi}{2} < \theta < \frac{5\pi}{2}$, so in the second equation on the left hand side we have to subtract $2\pi i$ to stay on this branch.

Euler knew that logarithms for all numbers were multivalued, even if a is a real number greater than zero: $\ln a = c$, then $\log a = c + 2k\pi i$. In addition, he showed that the logarithm of a complex number was a complex number. He did this by expressing

$a + bi = e^{x+iy}$, where $y = \arctan\frac{b}{a}$ and $x = \ln\sqrt{a^2 + b^2}$. Then

$\log(a + ib) = \ln\sqrt{a^2 + b^2} + i\tan^{-1}\frac{b}{a} + 2k\pi i$, which is a complex number for all $a + ib \neq 0$,

provided we define y to be $-\frac{\pi}{2}$ if $b < 0$ and $a = 0$, and $y = \frac{\pi}{2}$, if $b > 0$ and $a = 0$.

Euler extended his work on logarithms to all the elementary functions, that is, he showed that all the elementary transcendental functions have complex ranges if the domains are extended to the complex numbers.

In case of exponents, Euler first solved the vexing problem of evaluating i^i :

$i^i = \left(e^{i\frac{\pi}{2}+2k\pi i}\right)^i = e^{-\frac{\pi}{2}-2k\pi}$, for k any integer. So i^i has infinitely many real values. In 1748 he showed that any complex number of the form $(a + ib)^{c+id}$ could be expressed as p + iq , Boyer [70].

Whereas Euler produced many computational results about the complex numbers, another great eighteenth century mathematician, d'Alembert was less successful. d'Alembert attempted to show that algebraic operations on the complex numbers yielded complex numbers. In addition, he made several attempts to prove the fundamental theorem of algebra, and attempted to develop a calculus of the complex numbers, but did not have much success with any of these efforts, Boyer [70]. One success that is apparently due to d'Alembert is the use of the form a + ib for a complex number. Of course, d'Alembert did not use the symbol "i", (he used $\sqrt{-1}$), but the idea of real and imaginary parts is what is important.

The eighteenth century closed with the earliest inventions of what has become the modern complex plane. Bottazzini [75] cites an 1847 work by Cauchy that credits Henri Truel with the discovery of the complex plane. However, other authors, including Boyer [70], and Nahin [71], attribute the discovery to Casper Wessel in 1797, while Laugwitz [73] believes Gauss was the inventor. Nahin agrees that Gauss *could* have known about the complex plane as early as 1796. In any case, the full development and popularization of the idea of the complex plane had to await the next century, so we will defer discussion until we discuss period 3 in the next section. Finally, at the very end of this period, in 1799, Gauss showed how to prove the fundamental theorem of algebra.

In summary, the period between 1650 and 1800 was characterized (for our purposes) by the intense search for applications of calculus (the efforts were of course, very successful). This was also a period of distrust of geometry, since many flaws were discovered in the standard proofs of Euclidean geometry (the flaws consisted of statements that depend on diagrams) (Laugwitz, [73]). Thus, thanks in large part to the work of Euler, complex exponents, logarithms, and all the other complex transcendental functions were characterized and the ranges shown to be subsets of the complex numbers, but little progress was made on geometric questions until the very end of this period.

4.3.3 The Third Period – The Nineteenth Century

The third period of our study saw the development of complex numbers in four major areas. The development of the rudiments of the complex plane in the early part of the nineteenth century, the development of complex analysis, the development of the complex numbers as an algebraic structure (a field), and further development of the complex plane by Riemann in the 1850's. Perhaps more important than any of these developments was the growth in the general acceptance of complex numbers by the mathematics community. By the end of this period there were still reputable mathematicians who did not believe that complex numbers were correct, but these voices had become a small minority by the 1880's.

For our purposes, the details of the development of complex analysis and the advanced geometry of the complex plane are not particularly illuminating, so we will focus on the development of the complex plane (without the point at infinity) and the development of the complex numbers as a field.

As previously mentioned (in section 4.3.2), Cauchy attributes the development of the complex plane to Henri Truel in 1786, and it is possible, even likely, that Gauss thought of the complex plane by 1796. However, the first clear record of the plane is due to Casper Wessel in 1797. Wessel expressed complex numbers in the form a + ib, where a and b were real. Then he let a and b be the Cartesian coordinates of points in the plane: a was determined on the horizonal axis, and b was determined on the y axis. Of course, by 1797 the idea of representing negative numbers, then 0, then positive numbers in order on a line, due to Girard (1595 – 1632), and the idea of Cartesian coordinates due to Descartes were well known (Boyer, [70]).

Nevertheless, it is interesting that the idea of representing imaginary numbers on an axis perpendicular to the real axis was suggested by Wallis more than 100 years previously (Boyer, [70]). Apparently, Wallis could not bring himself to interpret the "+" in a + bi as simply the addition of Cartesian components. This is quite interesting, since as we saw in sections 4.2.5.2 and 4.2.5.3 it is actually incorrect to interpret "+" as addition of Cartesian components in the extended complex plane. Since at the time of Wallis the extended complex plane had not been invented, it would be interesting to investigate his thinking in much more detail. Probably the best explanation (of why Wallis did not invent the complex plane when he appears to have been so close) is that it was not until the work of Euler and d'Alembert that it became clear that all complex numbers could be expressed in the form a + ib, with a and b real.

In any case, we return to the development of the complex plane and the work of Wessel. Wessel was aware of the polar interpretation of complex numbers (due to Euler) and showed that this representation fit into the geometric model as complex numbers in polar coordinates. He used the notation used by modern electrical engineers for the polar representation: $a + ib = \sqrt{a^2 + b^2} \angle \tan^{-1}\left(\frac{b}{a}\right)$. This expression is read "the square root of a squared plus b squared *angle* arctan of b over a".

Wessel further argued the product of two directed line segments with polar angles θ and α had to have length equal to the product of the two segments, and polar angle $\theta + \alpha$. His reasoning was that the argument of the product had to differ from each angle by the other. With the new model, Wessel was able to derive DeMoivre's Theorem (see section 4.2.1) (Nahin, [71]).

Unfortunately, Wessel's work was not well publicized and the idea was apparently forgotten (Gauss, of course, did not forget). The complex plane was re-invented by M Buée in 1806, in a long philosophical treatise (Bottazzini, [75], Nahin, [71]). Robert Argand first printed his pamphlet on the complex plane in the same year, but it was not published until 1813 when the work found its way into the Annales de Mathematiques. Argand's work had been sent to the Annales by Jaque Francais, who obtained a copy of the pamphlet from Legendre. Francais pleaded for the author to come forward. Argand responded by publishing his pamphlet in 1814.

The publication of a geometric interpretation of the complex plane inspired much discussion about the nature of complex exponents. For example, Francais derived the result:

$\left(c\sqrt{-1}\right)^{d\sqrt{-1}} = e^{-d\frac{\pi}{2}}\left\{\cos[d\ln(c)] + \sqrt{-1}\sin[d\ln(c)]\right\}$. Of course, a result like this must surely have been known to Euler, if not Gauss.

Interestingly, the development of a geometric representation of the complex numbers did little, in the short term, to boost confidence in the existence of complex numbers. For example, Cauchy regarded complex numbers as "inexact", in effect, a "bookkeeping" device one used to do the work of two real equations in single calculations (Bottazzini, [75]). Thus, for Cauchy, the equation, $\cos(a+b)+i\sin(a+b)=(\cos a+i\sin a)(\cos b+i\sin b)$ was "inexact" and had no meaning. However, two real equations can be extracted from this equation, namely, $\cos(a+b)=\cos a\cos b-\sin a\sin b$ and $\sin(a+b)=\cos a\sin b+\sin a\cos b$, which Cauchy regarded as exact.

As already mentioned, during the eighteenth century and the first half of the nineteenth century geometry was held in suspicion, so it is perhaps not surprising that interest in the complex plane waned for a second time. The revival took place in France and Britain in 1828 with books by C.V. Mourey and John Warren , respectively (Nahin, [71]). In Germany the plane was again popularized by publication of a paper by Gauss in 1831, (Nahin, [71]) (this paper was published in 1832 according to Bottazinni, [75]).

Warren's book was a treatise on geometric representations of square roots of negative numbers and is alledged to have inspired William Hamilton to invent a better approach. Gauss's paper is thought to have been the publication of work he had done as early as 1796. In this paper he represents complex numbers in the form a + ib and as points in the plane. It is clear, that Gauss had fully accepted complex numbers by the time of the 1831 publication, but probably had accepted complex numbers much earlier. In 1796 he used the roots of $x^n - 1$ to construct a regular 17-gon (Fauvel and Gray, [74]).

In his construction he uses i for $\sqrt{-1}$, and recognized that the roots are given by

$$r = \cos\left(\frac{kP}{n}\right) + i\sin\left(\frac{kP}{n}\right), \quad k = 0, 1, 2, \cdots n-1, \text{ and } P = 2\pi.$$

In any case, Gauss's publication on complex numbers in 1831 also had little short term impact on the acceptance of complex numbers amongst the mathematics community. For example, in his 1831 book *On the Study and Difficulties of Mathematics*, De Morgan says the symbol $\sqrt{-1}$ is void of meaning, or self-contradictory and absurd. George Airy, who was astronomer royal from 1835 to 1881 expressed the view that he did not have the slightest confidence in results that used complex numbers (Nahin, [71]). In 1854, even algebraist and logician George Boole found $\sqrt{-1}$ to be uninterpretable in an investigation of the laws of thought.

In any case, William Hamilton finally did do "better" than John Warren and introduced his formal approach to complex numbers in 1835 (Nahin, [71] and Laugwitz, [73]. Bottazzini, [75] says this paper was published in 1837). This is the approach we have discussed under the name Cartesian vector representation in section 4.2.2, that is, complex numbers are represented by ordered pairs with two operations defined by $(a,b)+(c,d)=(a+c,b+d)$ and $(a,b)(c,d)=(ac-bd,bc+ad)$. Hamilton's work is of great significance theoretically, since he showed that the complex numbers are a field and therefore are internally consistent if any field is internally consistent. Since everyone assumed the real number field was internally consistent, there was no choice but to accept complex numbers as well.

Of course, this did not happen. In fact, Hamilton's work was largely ignored, even among those mathematicians who accepted complex numbers: most mathematicians using the complex numbers preferred the a + ib form or the polar form. Nevertheless, complex numbers were gradually accepted by most mathematicians. In 1849 Cauchy finally accepted complex numbers as vectors in the complex plane (Bottazzini, [75]).

We close our history of complex numbers with a brief mention of Georg Riemann. In his doctoral dissertation in 1851 Riemann introduced many of the advanced ideas of the extended complex plane, such as the point at infinity and Reimann surfaces. He developed his ideas further in the period 1851 – 1859. Riemann's work was very advanced and although deeply admired by Gauss, did not become well known until the end of the nineteenth century.

Concluding, in the third period (the nineteenth century) the geometry of the complex plane was gradually popularized, and the a + ib representation and the polar representation for complex numbers became the preferred representations. Hamilton introduced the Cartesian vector representation but it was not well received. Finally Riemann developed many of the modern ideas of the complex plane. In spite of the acceptance of complex numbers by major mathematicians such as Gauss, general acceptance of complex numbers was slow. Gradually acceptance grew until by the 1850's or so most of the mathematics community, at least grudgingly accepted complex numbers. Nevertheless, there were regions that heldout: thus, as late as the 1880's one top student at Cambridge lamented that $\sqrt{-1}$ was still widely regarded as suspect by his teachers (Nahin, [71]).

4.3.4 The History of Complex Numbers - Summary and Conclusion

We have seen that complex numbers were ignored until the discovery of formulas for cubic and quartic equations. These formulas sometimes result in real solutions that are expressed in terms of complex numbers. Thus, the need to learn how to manipulate complex numbers arose in order to extract the real solutions. Nevertheless, acceptance of complex numbers took a long time. In the first period (antiquity – 1650) the search for a geometric representation of the complex numbers was substantial, but unsuccessful.

In the second period (1650 – 1800), work on a geometric representation continued, particularly by Wallis. Newton made an intense effort to prove the fundamental theorem of algebra, and Euler extended all the elementary functions to functions defined on the complex numbers. Euler also defined complex logarithms and exponents. d'Alembert conjectured that every complex number could be expressed in the form a + ib, although he was unable to show this. At the end of this period, Wessel (and probably Gauss) discovered how to represent complex numbers in a plane, and defined the basic operations on complex numbers in the plane.

In the third period (nineteenth century) the complex plane was rediscovered and gradually popularized by the time of Riemann in the 1850's. The a + ib reprentation was in common use in the modern form (i instead of $\sqrt{-1}$). Hamilton introduced what we have designated the Cartesian vector representation in 1835, but this development had little impact on acceptance of complex numbers. Finally the idea of the complex plane was greatly extended by Riemann in the 1850's.

Having briefly reviewed the historical development of complex numbers, we now discuss some of the conclusions we can make about what was hard to understand for the mathematics community and why.

First, it is obvious that there was an enormous reluctance to accept complex numbers, even as late as Gauss's 1831 paper re-introducing the complex plane. In this paper, Gauss expresses the view that the problem was the use of terminology such as "impossible" or "imaginary" which has such strong connotations in ordinary language (Nahim, [71]). In our research of the history of complex numbers (entirely from secondary sources) we have culled no clear insights into why complex numbers were so hard to accept. Apparently, the difficulty is entirely due to the "knowledge" that negative numbers do not have square roots. Acceptance of complex numbers was further delayed by the aversion to geometry that pervaded the second period and the first part of third period: even the development of good geometric representations of the complex numbers did not immediately inspire wide acceptance of complex numbers. In any case, the difficulty the mathematics community had accepting complex numbers is quite sobering.

In section 4.2 and subsequent sub-sections, we discussed many problems of understanding the details of various representations. We now turn our attention to what we can learn about such questions from the historical record that we have recounted.

We have previously alluded to the fact that Wallis appears to have struggled with the "+" in a + ib, however in our research of the history of complex numbers we found no other evidence of a mathematician having difficulties with the "mechanics" of complex numbers. According to Nahin [71], even an outspoken critic such as De Morgan was completely fluent with all the calculations being performed in his day (1830's). It is true that for example, Cardano was unable to solve complicated simplification problems, such as the difference between cube roots of complex conjugates, but this hardly shows a lack of understanding of the basic "mechanics" of complex numbers discussed in section 4.2. Rather, this shows Cardano must have had a good understanding to conceive of the problem.

On this note we move on to the methodology used to collect the data for our study.

Chapter 5

Methodology of Data Collection and Analysis

5.1 Method of Data Collection

We collected data primarily from attending the three classes studied, from informal study sessions with students and from clinical interviews. We have divided this section into three subsections, one section for each of the three classes studied.

5.1.1 Data Collection: Pilot Project - Class 1

To get started with our research goals we began with a pilot project in the summer semester of 1996, at Simon Fraser University. Subjects were recruited from the regular Math 322 class (Introduction to Complex Analysis). This class met three times per week for lectures, but did not have a tutorial session (somewhat unusual at Simon Fraser University). There were about sixty students enrolled at the beginning of the class, and lectures were held in a small size (for Simon Fraser University) lecture hall. There was a teaching assistant available to mark homework, but not for tutoring.

The instructor used two overhead projectors, but no microphone. The textbook for the course was *Introduction to Complex Analysis*, by H. A. Priestley [68]. Evaluation of student performance was done with eight homework assignments and 3 one hour tests. Homework assignments were taken from the problems in Priestly.

The material covered in the course consisted of the first eight chapters of Priestly. These topics include complex numbers and basic topology of the complex plane, holomorphic functions and power series, preparation for Cauchy's theorem, Cauchy's theorem, logarithms, consequences of Cauchy's theorem, singularities and multivalued functions, residues, and applications. The instructor followed the material covered in the textbook closely, in most cases, but frequently used his own presentation.

There was no use of computers during this course, although the instructor gave a demonstration of Maple in the last week of classes.

We recruited all our subjects for the pilot study from this class. At the second lecture we gave a brief description of the study and asked for volunteers. We circulated an information sheet and consent form (see appendix 1) to all of the students present. As compensation, we offered to help students with the material as much as they liked. We were available seven hours per week for this purpose.

Six students volunteered for the pilot study. We conducted two separate sets of interviews of about one hour duration each. The first one was done during week eight of classes, and the second was done after the final one hour test. A total of seven interviews were conducted. In addition, we gave extensive help to all six students.

Finally, we collected data by attending all but two of the lectures, including one of the one hour tests. We tape recorded the first few lectures, and took careful lecture notes, including comments about teaching methodology, layout of the room and lecture equipment, as well as noting student responses, such as questions, corrections to mistakes by the lecturer, and answers to questions by the lecturer.

All interviews were qualitative in nature, with significant questioning from us. In designing clinical interviews we followed the recommendations of Ginsburg [77], Lincoln and Guba [78], Howe and Eisenhart [79], Asiala *et al* [46]. Interviews consisted of students working through the problems on a prepared worksheet, while explaining their thoughts orally. Interviews were tape recorded. Specific questions can be found in appendix 3, questions and solutions that we used in our analysis in chapter 6 are in appendix 4, and two samples of complete interview transcripts (with worksheets) can be found in appendix 5. We did not hesitate to ask students what they were thinking about, and we freely helped them if they were stuck.

We achieved four important objectives with our pilot study that helped us with our main study:

1. We determined a large number of difficulties that students have with learning complex analysis.

2. We were able to get some information about what level of question worked well in interviews.

3. We collected specific data on how the instruction and textbook were influencing student conceptions of complex numbers.

4. We gained important interviewing experience.

We obtained enough good data from our pilot project to include this data in the overall results. For this purpose, in later chapters, we refer to the pilot project as class 1, for brevity.

5.1.2 Data Collection: Main Study - Class 2

Our main study took place in the Fall of 1996, at Simon Fraser University. The class was, again, Math 322, Introduction to Complex Analysis, but the textbook was changed to *Complex Variables and Applications*, by Churchill and Brown [69]. This time there were about forty five students enrolled. This class met three times per week for lectures, and there was a tutorial once per week (conducted by the instructor). Lectures and tutorials were held in a large size classroom, that had no windows and had poor acoustics.

Evaluation of student performance was done with weekly homework assignments, 2 one hour tests, and a final examination. Homework assignments were taken from the problems in Churchill and Brown. A teaching assistant was available for marking only. As in the previous class, computers were not used. The instructor did present graphs done with MS Excel on one or two occasions.

The material covered in the course consisted of the first sixty sections of Churchill and Brown. The topics in these sections are arranged under the headings: complex numbers, analytic functions, elementary functions, integrals, series, and, residues and poles. In addition, there are many applications. During lectures, the instructor of class 2 followed the material covered, and the presentation given in the textbook very closely. Tutorials were used to discuss problems or concepts that were not covered in the text. During tutorials, students were asked at random to do problems at the board, in front of the class.

We recruited six participants from class 2. At the fourth lecture we gave a brief description of the study and asked for volunteers. We circulated an information sheet and consent forms (see appendix 1 and 2) to the entire class. As compensation, we offered to pay student $20 per interview, and we offered to help students, with the material, as much as they liked. We were available seven hours per week for this purpose.

We conducted six separate sets of interviews of about one hour duration each.

We began interviewing in the third week, and subsequent interviews were held approximately every two weeks after that, up to the last week of classes. We conducted a total of thirty interviews of these six class 2 students (two did all six interviews). In addition, we gave extensive help to three of the students.

Finally, we collected data, by attending all but four of the lectures, not including the lectures turned over to taking tests. In addition, we attended all but one of the tutorials. As with our pilot project, we took careful lecture notes, including comments about teaching methodology, layout of the room and lecture equipment, as well as, noting student responses, such as questions, corrections to mistakes by the lecturer, and answers to questions by the lecturer.

Since we were successful using interviews in our pilot project, we chose to continue using this method of collecting data. We used the same format that we used in the pilot study: a prepared worksheet with several questions (usually more than the student could possibly do in one hour), in three distinct categories, together with audio taping of students while they explained what they were thinking. As with our pilot project, we did not hesitate to ask students what they were thinking about, and we freely helped them if they were stuck. Specific questions can be found in appendix 3, questions and solutions that we used in our analysis in chapter 6 are in appendix 4, and two samples of complete interview transcripts (with worksheets) can be found in appendix 5.

5.1.3 Data Collection: Main Study - Class 3

The second part of our main study took place in the Spring of 1998, at the University of British Columbia (UBC). The class was Math 300, Introduction to Complex Analysis, and the textbook was *Fundamentals of Complex Analysis for Mathematics, Science, and Engineering* by E. B. Saff and A. D. Snider [65]. Math 300 is normally a year long course at UBC, but it is also possible to take it as a one semester course. We studied the one semester version.

Class 3 had about forty five students enrolled. The class met three times per week for lectures. There were no tutorials, but there was an on-line discussion group for the class. Lectures were held in a large, well lit class room. The instructor used the chalk board.

Evaluation of student performance in class 3 was done with weekly homework assignments, 2 one hour tests, and a final examination. Homework assignments were taken from the problems in Saff and Snider. A teaching assistant was available for marking only.

The material normally covered in the one semester version of Math 300 consists of the first six chapters of Saff and Snider: Complex numbers, analytic functions, elementary functions, complex integration, series, and residue theory. During lectures, the class 3 instructor followed the material covered, and the presentation given in the textbook closely.

We recruited nine students from class 3. At the second lecture we gave a brief description of the study and asked for volunteers. We circulated an information sheet and consent form to all students in the class. As compensation, we offered to pay students $20 per interview, and we offered to help students, with the material, as much as they liked. We were available six hours per week for this purpose.

Our plan for this part of the study was to obtain more detailed data on the topics in the early part of the course. We wanted to gather more data on our theme of multirepresentations of complex numbers. We conducted two separate sets of interviews of about one hour duration each. The first interview was conducted at the end of the third week of classes, and the second interview was done at the end of the fifth week of classes. We conducted a total of seventeen interviews of these nine class 3 students. In addition, we gave occasional help to four of the students.

Finally, we collected data by attending all but one of the lectures, until the first one hour test . As with the previous two classes, we took careful lecture notes, including comments about teaching methodology, layout of the room and lecture equipment, as well as noting student responses, such as questions, corrections to mistakes by the lecturer, and answers to questions by the lecturer.

Since our objective for studying class 3 was to focus on problems students have understanding complex numbers, limits and continuity, we dropped general questions on ethno-graphics and affect, and also the true false format, from the two interviews we did in class 3. We did retain use of a prepared worksheet with several questions (more than we expected the students to do in one hour), together with audio taping of students while they explained what they were thinking. As with class 1 and class 2, we did not hesitate to ask students what they were thinking about, and we freely helped them if they were stuck. In addition, we challenged students more aggressively than in the previous two classes, because we were interested in testing the robustness of their beliefs. Specific questions can be found in appendix 3, questions and solutions that we used in our analysis in chapter 6 are in appendix 4, and two samples of complete interview transcripts (with worksheets) can be found in appendix 5.

5.2 Method of Data Analysis

In this study we have allowed the data to determine the theoretical organization of our analysis. Hazzan [80] used this method of analysis to analyze data collected while studying students of abstract algebra. As Hazzan points out, allowing the data to shape the theoretical framework is especially suitable for research done in a new field. Hazzan found that a broad educational theme, namely *Reduction of the Level of Abstraction*, could explain most of their data.

We have tried to explain our data in terms of one content related theme, namely, Multiple Representations. The complex numbers have a number of representations: vector, algebraic, polar, and symbolic. Understanding these representations, and developing the ability to fluently shift between them, is an important objective for students in the early part of the course. In addition, we have included a final section illustrating student problems with later material due to uneven skill with basic material.

The restriction of our analysis of the data to the one theme of multirepresentations has meant that much of the data was not used. Several other themes emerged from the portion of the data not used in this study. Other themes that arose from the data, such as "thinking real, doing complex" and "multivaluedness" were especially interesting, but had to be omitted for reasons of clarity of analysis and to keep this thesis at a reasonable length.

Of the many possible themes that arose from our study of the data we chose to report on the single theme of *multirepresentations* for three reasons: 1. The single theme with the most data in support of that theme was multirepresentations. 2. Since complex analysis is an almost entirely unstudied area of research in mathematics education, it made sense to us to focus on the beginning portions of the course. We believe the first stage of opening a new area of research in mathematics education is building a solid body of literature on the elementary aspects of complex analysis. Once some of the difficulties with elementary material have been identified and studied, and there is some consensus by researchers in the field about what direction to take, then it will be appropriate to begin reporting on research on more advanced themes. 3. We believed that it was best to focus on one theme given that we were reporting on research in a new area of mathematics education.

5.3 Summary

We collected data from three beginning complex analysis classes held at Simon Fraser University and the University of British Columbia. Data was collected by attending classes, from individual study sessions with students, and from clinical interviews.

We allowed the data to determine the theoretical organization of our analysis. Of the several themes that emerged from our data, we chose to report on the data and analysis pertaining to the single theme of multirepresentations.

Chapter 6

Results and Analysis

In this chapter we analyze the data that pertains to our theme of Multiple Representations of Complex Numbers. We begin with an overview of the subjects covered in several sections. Next we analyze our data on several topics in sections devoted to each topic. The general format of these sections is to present the questions we asked, present data, usually in tables, record interview excerpts where they are illuminating, and then proceed with an analysis of the results. Solutions to the questions we discuss in this chapter can be found in Appendix 4.

6.1 Introduction

We collected data on four representations of complex numbers. We have discussed these representations at length in Chapter 4, but we give a brief review and discuss some key points here.

The first three of these representations emphasize complex numbers as the range of a function of two real variables. These are: the vector form, $z = (x, y)$, the polar representation of complex numbers, $z = re^{i\theta}$, and the algebraic extension notation, $z = x + iy$. The vector representation clearly emphasizes geometric relationships. The polar representation is very useful partly because the inherent multivaluedness of the complex numbers is emphasized in a natural way in the polar format. The algebraic extension representation is used mainly as a vector representation (at least it was in the classes we studied), so the status of this form is less clear than the other forms. The fourth representation, called the symbolic form in this thesis, emphasizes the relationships between complex numbers and properties of complex numbers, as opposed to the construction of the complex numbers from the real numbers.

An important aspect of the vector, polar, and symbolic representations is that they have algebraic and geometric interpretations. Thus, we can think of addition of vectors in the $z = (x, y)$ representation in a formal way (algebraic view) or we can think of the sum of two vectors as the diagonal of the parallelogram formed by the two vectors (geometric view). In many cases, the problems in this study can be solved by drawing a rough diagram that illustrates the important relationships given in each problem, and then guessing (perhaps with minimal calculation) the exact answer from the diagram. We have given this solution strategy the name "geometric methods". A example of this strategy is as follows: Find all points in the complex plane that satisfy $|z - 2| = |z|$. If we draw a picture with the points 0 and 2 marked, and we note that geometrically this equation means that z has to be equidistant from 0 and 2, we can "guess" that the answer is the set of all points such that $x = 1$. Of course, this problem can be solved algebraically too: substitute $z = (x, y)$ or $z = x + iy$, and apply the defintion of the modulus to get a real number equation in x and y.

Since an important part of our analysis is concerned with shifting from one representation to another, we briefly describe how to do this. The shift between the vector representation and algebraic representation is immediate by $(x, y) \leftrightarrow x + iy$. To shift between the vector or algebraic representation and the polar representation amounts to shifting between Cartesian and polar coordinates: $x = r \cos \theta$ and $y = r \sin \theta$, and the inverse transformation, $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan\frac{y}{x}$. To shift from the symbolic representation to one of the other representations simply replace z, $\bar{z}$, etc with the representation that is preferred. Shifting from the vector, polar, or algebraic representation to the symbolic representation is more complicated, but can at least be done in a formal way by the substitutions $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, or $r = |z|$ and

$$\theta = \frac{\log\left(\frac{z}{|z|}\right)}{i}.$$

The contrast between the usefulness of different representations in complex analysis and real variable vector calculus ($R^2 \to R^2$) is profound. In real variable vector calculus we sometimes use a change of coordinates to make a problem easier, but there are no underlying structures that are conveniently represented by one representation as opposed to another. Hence the problem of multiple representations is largely absent in real vector analysis. We found students had many difficulties with multivaluedness and changing from one representation to another. In addition, some students struggled with more advanced material, because of problems with basic material, so we have included a separate section called Uneven Skill with Basic Material Affects Later Work that contains data and analysis of problems with basic calculations.

6.2 General Considerations

As we have already discussed, the complex numbers have several representations that are very useful. Students must gain some fluency with the different representations, their properties, and domains of utility if they wish to master complex analysis. We found that the task of learning the different representations, and when to use them, was largely taken for granted during instruction of the three classes we studied, but that students mostly struggled, using only one or two representations. We have organized our data in four sections: Shifting Representations, i is a Unit Vector, Basic Facts and Calculations, and Uneven Skill with Basic Material Affects Later Work.

Section 6.3 on Shifting Representations includes all the data and analysis we have about students' ability to decide on an appropriate representation of the complex numbers for a given problem. In this section we are not too concerned with how well students were able to use a particular representation. In other words, we have separated the process of *choosing* an efficient representation, from the process of *using* the chosen representation.

We collected significant data that strongly suggests that many students view the imaginary number i as a unit vector. We discuss this phenomena in section 6.4 on i is a Unit Vector.

Basic Facts and Calculations (section 6.5) includes some of the problems we found that students have with basic facts , such as, "Can you use the quadratic formula?", and "Why can you cancel a common factor from the numerator and denominator of a fraction?". In addition, we have gathered into this section all the results we have on the problems that students had with basic calculations.

Section 6.6 titled "Uneven Skill with Basic Material Affects Later Work" consists of our data and analysis of three problems of an advanced nature that were difficult for students because they did not have sufficient grasp of basic material.

6.3 Shifting Representations

The first group in our Multiple Representations theme is shifting representations. We found that some of the students studied did not make good decisions about when to shift representations, and that of those students who did make reasonable decisions, many could not bring their decisions to a correct conclusion.

6.3.1 Interview Questions and Results

To study students' ability and willingness to change we asked students in classes 1 and 2, a series of questions designed to see how long they would persist with the $z = x + iy$ or $z = (x, y)$ formulations. In particular, by this we mean persist with the algorithm for converting the form $\frac{a+ib}{c+id}$ to the standard forms $x + iy$ or $re^{i\theta}$.

The questions were as follows (these questions were in interview #1, for both classes):Simplify the following, i.e., express them as $a + ib$, or as $re^{i\theta}$ (whichever you prefer).1. $\frac{2+i}{2}$ 2. $\frac{2}{1+i}$ 3. $\frac{2+2i}{1+i}$ 4. $\frac{a+ib}{a-ib}$ 5. $\frac{|a+ib|}{a+ib}$ 6. $\frac{-2+2i}{(1+i)^3}$

7. $-\frac{8}{(\sqrt{3}+i)^6}$ 8. $\frac{2\cdot(1+i)^4}{(-2\cdot\sin 15^0 + 2\cdot i\cos 15^0)^5}$ (Class 1)

8. $\frac{2(1+i)^4}{(-2\sin(\frac{\pi}{12}) + 2i\cos(\frac{\pi}{12}))^5}$ (Class 2)

Solutions to these questions can be found in appendix 4. Five and six students were questioned, respectively, in classes 1 and 2. We were primarily interested in finding at what stage the students would shift from the algebraic form, $z = x + iy$, or

Cartesian form, z = (x, y) to some other form, but we also wanted to study what problems students had working with whatever representation they chose.

We expected that students would probably shift to the polar form, $z = re^{i\theta}$ at question 6, and that they would definitely attempt to do question 8 using the polar form. In addition, questions 4, 5 can be done symbolically or using geometric methods, and question 3 can be done by cancellation. The results are tabulated in Table 6.1. The columns in the table are the question numbers, and the entries in the table are the number of student who attempted to use the z = x + iy or z = (x, y) form (using the method of realizing the denominator[1]) to simplify the given expression.

Table 6.1

Number of Students who Attempted Questions Using z = x + iy

Class	1.	2.	3.	4.	5.	6.	7.	8.
1	5	5	3	3	3	1	1	0
2	6	6	6	6	6	4	3	1

Table 6.1 shows a clear trend by respondents to shift to other methods to do the last few questions.[2] In class 1 students began to shift to other methods as early as question 3. Class 2 students did not shift to other methods until question 6. It is interesting that question 6, appears to be a turning point for class 1 students, as well: two class 1 students shifted methods at question 6. Thus, all but one class 1 student had

[1] The term "realizing" the denominator is not in common use in complex analysis, but it should be. The term that is usually used is "rationalizing" the denominator. Multiplying a complex number by its complex conjugate always produces a real number, but does not necessarily produce a rational number, so the accurate term is "realizing" the denominator. For example, $(e + i\pi)(e - i\pi) = e^2 + \pi^2$, which is not rational.

[2] One student in class 2 used the vector form, z = (x, y), for questions 3-8. This has been counted as the z = x + iy form for the present purposes.

shifted methods by question 6, whereas, it was not until question 8, that all but one class 2 student shifted.

6.3.2 Analysis of Data on Shifting Representations

Our data can be interpreted to indicate that students in these two classes were not able to access symbolic or geometric methods, in effect, the shifts were almost exclusively between algebraic or Cartesian vector methods and polar vector methods. For example, every student who attempted to simplify $\frac{|a+ib|}{a+ib}$, (question 5) using the $z = x + iy$ or $z = (x, y)$ representation, worked as follows:

$$\frac{|a+ib|}{a+ib} = \frac{\sqrt{a^2+b^2}(a-ib)}{(a+ib)(a-ib)} = \frac{a\sqrt{a^2+b^2}}{a^2+b^2} - \frac{ib\sqrt{a^2+b^2}}{a^2+b^2}. \qquad (6.1)$$

Some students made mistakes, and some left $\sqrt{a^2+b^2}$ as $|a + ib|$, but none of the students cancelled the common factor of $\sqrt{a^2+b^2}$. This last fact probably comes from being trained in high school that (6.1) is in "standard form". Two students (one in each class) noticed that $\frac{|a+ib|}{a+ib} = \frac{|z|}{z}$, but didn't pursue this idea further. This was a good observation, since we can save work: $\frac{|a+ib|}{a+ib} = \frac{|z|}{z} = \frac{\bar{z}}{|z|} = \frac{a-ib}{\sqrt{a^2+b^2}}$. Of course, a geometric solution was also possible, but no one attempted it. Thus, the shift in representations, in these questions, that was made by all but one student was from algebraic or Cartesian vector form to polar vector form.

Table 6.1 shows a strong trend by students to shift methods. The strong differences between the two classes, should not be taken too seriously, because class 2 students were asked these question in the third week of classes, whereas class 1 students were asked between the eighth and twelfth weeks of classes. (Analysis of student work sheets reveals some evidence that students became more comfortable with the polar form as the course progressed.) Furthermore, one class 1 student shifted methods by canceling the common factor of 1 + i from the numerator and denominator of question 3.

We believe that the common factor of (1 + i) in the numerator and denominator of question 3 $\left(\frac{2+2i}{1+i}\right)$ suggested to two students that they shift. One appeared to notice that the numerator and denominator of this number are parallel, as vectors, in effect having the same argument. (We would classify the observation of "parallel numbers" as a geometric method of solution, so this is the one instance of a geometric approach that we observed for these questions.) The arguments of the numerator and denominator were the key for these two students in questions 4 and 5 $\left(4.\ \frac{a+ib}{a-ib},\quad 5.\ \frac{|a+ib|}{a+ib}\right)$ as well: both students quickly realized that the moduli of the numerator and denominator would cancel (in polar form), and concentrated their attention on how to find the final argument (having established that the modulus of the number is one). Students who did not shift until question 6 $\left(\frac{-2+2i}{(1+i)^3}\right)$ appeared to be motivated by the desire to save work. None of the students who shifted at question 6 saw that the argument of the numerator and denominator were related and that this fact could be exploited to save work. Instead, they appeared to base their shift on the understanding that the polar form was easiest for a division problem involving an expression with an exponent. Thus, we have a sequence of increasingly sophisticated thinking on these problems, as follows:

Stage 1. Multiply the numerator and denominator by the complex conjugate of the denominator. In other words, the "realizing" algorithm is routinely applied.

Stage 2. Cartesian form is too hard, the polar form should be easier for a division problem. In this stage the Cartesian form is recognized as too difficult, but the polar form is still applied routinely.

Stage 3. The arguments of the numerator and denominator appear to be related (either equal or negatives), so cancellation or polar form should be easier.

Stage 4. (For question 6) The argument of -1 + i is 3 times the argument of 1 + i, but the 1 + i is cubed, so the arguments will cancel, so we only need to consider the moduli. But these are equal, so the answer is one. In stage 4, stage 3 thinking is combined with a good grasp of the geometry of the plane, so that the exact relationships between the arguments of the numerator and denominator can be visualized, and used to assess whether switching to polar form is likely to be helpful.

In question 7 $\left(-\frac{8}{(\sqrt{3}+i)^6}\right)$, all but one of the students who used the polar form recognized quickly that the argument of the denominator was π. Nonetheless, we do not think this should be taken as evidence of stage 4 thinking, because all the students appeared to stumble onto this result: they appeared to be motivated to use the polar method because the Cartesian method looks very tedious. Specifically, the exponent of 6 in the denominator strongly suggests that Cartesian methods will be difficult, whereas exponents are easily handled in the polar representation. Thus, our data gives no indication that students were doing anything more than applying the rules of the polar representation in a routine way, which is stage 2 thinking. In the case of question 7, it is not entirely clear how stage 4 thinking would manifest itself, in effect, how stage 4 thinking would be clearly different from stage 2 thinking. The problem is that in question

7 the Cartesian method is much harder than the polar form, so the assessment process, in effect, the process of deciding to shift to polar form is very brief. Our data is not detailed enough to reveal which students just saw the exponent of 6 in the denominator and shifted (stage 2 thinking), and which students looked ahead, visualized where $\sqrt{3}+i$ is in the complex plane, looked for a geometric solution, etc. (stage 4 thinking).

Lesh, Post, and Behr [81] have identified three qualities of understanding the idea "1/3", that are relevent to the present discussion. Before listing these qualities, we note that they are not sufficient to explain our data, and that the list needs to have one additional quality added, as explained later in this section. Here are the three qualities identified by Lesh, Post and Behr:

1. Recognition (by the student) of the idea embedded in a variety of qualitatively different representational systems

2. The ability to flexibly manipulate the idea within given representational systems

3. The ability to accurately translate the idea from one system to another.

Our interpretation of our data indicates that the students studied could accurately use the algebraic and polar vector forms (although the polar form was hard for some students), and most could accurately represent a complex fraction in either of these forms. Thus, criteria 1 and 2 were met in a limited way (ignoring the geometric aspects of each of the vector representations and ignoring the symbolic representation completely). The main difficulty students had shifting from Cartesian to polar form was not knowing basic trigonometry, such as what arguments are possible if the cosine is $\sqrt{3}/2$, or realizing that arctan (b/a) is not single valued, so that this expression does not uniquely specify the argument of a + ib. Thus, criterium 3 was not as clearly met (as criteria 1 and 2) by most of the students studied.

Lesh, Post and Behr were primarily concerned with students in primary school, so the idea of shifting representation may be too sophisticated for such students. Furthermore, a fraction such as 1/3 is already simplified, whereas, complex fractions, by convention, are not in simplest form unless the denominator is real. Therefore, complex fractions are inherently more complicated than rational numbers. Thus, it would be surprising if the Lesh, Post, and Behr criteria were adequate to describe understanding of complex fractions.

More specifically, we have been arguing (in fact, the whole set of questions was predicated on the assumption) that there is a fourth criterion that needs to be added to the Lesh, Post, and Behr list, namely the ability to recognize when it would save work to shift representations. Other authors have noted that shifting representation is an important aspect of understanding representations in mathematics. For example, Dreyfus [82] identifies four stages in the learning process of a general representation in mathematics: 1. Using a single representation. 2. Using more than one representation in parallel. 3. Making links between parallel representations. 4. Integrating representations and flexible switching between them.

It is clear from our data that if we confine our attention to Cartesian and polar forms that students are able to work with the Cartesian form, the polar form, and can even translate from one form to the other (reasonably well), but that 5 out of 11 students questioned did a question as hard (in Cartesian form) as question 6 $\left(\frac{-2+2i}{(1+i)^3}\right)$, using Cartesian methods. This indicates that the ability to decide when to shift representations is an important and separate aspect of our conception of complex fractions, and should not be included as one of the three criteria in the Lesh, Post, and Behr definition.

Thus, we are suggesting that a fourth criterion needs to be added to the Lesh, Post, and Behr definition of understanding a fraction, *the ability to decide when to shift representations*, and that the four stages of thinking about shifting discussed above, are different levels of ability within this fourth criterion. Since this section is about shifting representations we are only concerned with this fourth criterion, and will now analyze the first three stages of thinking we have identified in more detail (recall that we have no data showing stage four thinking).

In the analysis below we are primarily concerned with identifying evidence of *process - object* understanding (using the Sfard-Linchevski-Kieren reification model discussed in section 2.1.2.5) at each of the first three stages of thinking we have identified for shifting representations. However, where possible, we have included a possible characterization of the simplification problem into the *action - process - object* understanding (using the APOS model discussed in section 2.1.2.5). Also, recall that we conjectured in sections 4.2.1.3, 4.2.2.3, 4.2.3.3, 4.2.4.3 about what activities would indicate process and object understandings. Since the task being considered (simplification of a complex fraction) is relatively simple, we expect the differences between these levels of understanding to be subtle. From our data, we have identified general characteristics of process and object understanding (reification) and action,

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Reification

Process understanding - concerned with the details of simplification. Uses the realizing algorithm routinely. Applies the polar form routinely. Multiplies (a + ib)(a - ib) in the denominator (i.e., does not recognize or is unable to apply the fact that the whole point of multiplying by the complex conjugate of the denominator is that the product will be the modulus squared, which is a real number).

Object understanding - Uses prior knowledge, such as, $(a + ib)(a - ib) = a^2 + b^2$. Looks for a better way to do the problem, such as, cancellation. Uses geometry to simplify the algebra where possible. Is able to use the polar form without the formalism, e.g., can simplify an expression such as $(\sqrt{3} + i)^6$ without converting to the form $r^6e^{6i\theta}$ explicitly. Generally, indicates the ability to see simplification of a complex fraction as achievable in several ways, and shows evidence of consideration of more than one method.

APOS

Action understanding - Concerned with the details of the realizing algorithm for each problem. Applies algorithms routinely and apparently without thought of other possibilities (absence of evidence of a thought process leading to a decision to perform the calculations that were undertaken by the student).

Process understanding - Shifts to the polar representation at some point, but is still primarily concerned with the details of the simplification. Possibly uses some "short cuts", e.g., uses $(a + ib)(a - ib) = a^2 + b^2$ without calculation. Generally fluent technique, i.e., is able to do the simplification correctly with few obstacles. Shows that they are doing the calculations by decision, rather than because "it is what you do in this kind of problem".

Object understanding -- Uses prior knowledge freely. Indicates consideration of more than one method of solution. Uses geometry to aid calculations. Freely modifies realizing algorithm or polar method to save work.

Box 6.1

Descriptions of Types of Understanding of the Simplification Problem in Terms of the Sfard-Linchevski-Kieren and APOS Models of Understanding.

process, and object understanding (interiorization and encapsulation in APOS) in box 6.1 (more details of these types of understanding are included in the analysis below).

Although in the description above we have not described an object understanding in reification and APOS in exactly the same way, as previously discussed (section 2.1.2.5) we are assuming these understandings to be essentially the same. Hence, in the analysis below we have not specifically distinguished between the two models when discussing an object understanding. We also have not specified which model we mean when we refer to a process understanding, even though this concept is not quite the same in the two models. Instead we refer to an action understanding in the APOS model when needed.

Thus far, we have identified four stages of thinking about shifting representations. As well, we have attempted to identify, from our data, what activities and student thinking correspond to *process* and *object* understanding in the Sfard-Linchevski-Kieren reification model, and what activities and thinking correspond to *action, process,* and *object* understanding in the APOS model. It is important to recognize that we expect to see all levels of understanding in either model, at each of the four stages we have identified. This is because at each of the four stages there are several concepts to understand, and different levels of understanding are possible depending on whether the reification or APOS models is used to describe understanding at each stage. In the analysis that follows we have generally used reification, but sometimes we are able to separate *process* understanding in reification, into *action* and *process* in the APOS model.

Stage 1

Stage 1 thinking of *shifting representations* is "multiply the numerator and denominator by the complex conjugate of the denominator" (referred to as the realizing algorithm, or just the algorithm in the following discussion). This stage is actually a null stage: a student who uses this algorithm for putting a complex fraction in standard form, is at best demonstrating that they have the ability described in criterion 2 of the Lest, Post, and Behr scheme (ability to manipulate a single representation), and at worst is showing no ability to judge when to shift. Our data shows a range of thinking by students when they applied the standard algorithm, so that, although table 6.1 shows rather distinct shift points, student thinking was more complex.

For example, none of the students interviewed did question 1 $\left(\frac{2+i}{2}\right)$ using the algorithm: in one way or another, they all divided 2 into 2 and 2 into 1 to get 1 + 1/2 i, as opposed to applying the algorithm by multiplying top and bottom by 2 - 0i. Thus, at least all of the students recognized that they did not have to apply the algorithm unless the denominator was complex. One student had doubts about using the algorithm on question 4 and 6, but could not think of anyhing else to do. Another student treated each question as a multiplication problem. Thus, for example, they did question 3 essentially, as follows: $\frac{2+2i}{1+i} = (2,2)(1,1)^{-1} = (2,2)(\frac{1}{2},\frac{-1}{2}) = (1+1,0) = 2$. In this calculation the realizing algorithm is hidden in the formula for the inverse that this student has used, so we have not counted this as a shift in representations. Nevertheless, this student is thinking about this problem a little differently than a student who routinely applies the algorithm, because using the formula for the inverse is slightly easier than applying the algorithm (using the formula for the inverse avoids having to multiply the denominator by its complex conjugate). Actually, almost all of the students who applied the realizing algorithm in a routine way, also knew that the final denominator would work out to the sum of the squares of the real and imaginary parts of the original denominator, showing that students were attempting to be as efficient as possible.

Thus, even though most students applied the realizing algorithm to questions 2 to 5, some wondered if there was not a better way, and almost everyone incorporated some amount of prior knowledge in the form of other formulas, to make the method simpler.

We think that applying prior knowledge to make a procedure more efficient shows that these students were well passed having an action understanding (APOS). In fact, we believe these students either had or were well on their way to obtaining an *object* understanding of the realizing algorithm for simplification of complex fractions. By an object understanding of the realizing algorithm we mean the ability to think about the algorithm as a whole, recognize that the algorithm is inefficient or not the best way to do a particular problem, improve the algorithm, or compare the algorithm with other simplification strategies (such as cancellation). We believe that an object understanding of the methods of each representation is necessary for efficient shifts to occur, because students need to have an overview of the problem under consideration (irrespective of how it has been posed), and assess the merits of each representation, before choosing a simplification strategy. Thus, stage 1 thinking ranges from routine application of the simplification algorithm (action understanding in APOS) to full object understanding of the simplification algorithm in Cartesian vector form (compare with box 6.1). Within the context of the classes studied, it is hard to imagine a student acquiring an object understanding of the Cartesian representation without learning something about the polar representation (so that they would have moved to stage 2 thinking in our scheme), but since it is possible we include the possibility of an object understanding of the realizing algorithm in stage 1. Most of the students we studied appeared to either have a full object understanding of the realizing algorithm, or be close to obtaining such an understanding. Certainly, none of the students studied was thinking at the action level of understanding (APOS).

Stage 2

We have described stage 2 thinking of *shifting representations* as "the Cartesian form is too hard, the polar form should be easier for a division problem". As previously mentioned, this type of thinking appeared to be the justification for shifts by all students who shifted at question 6 ($\frac{-2+2i}{(1+i)^3}$). We have already noted that students made a number of minor mistakes when using the polar representation, such as not defining a branch of $\tan^{-1}\theta$, and when applying basic trigonometry. Thus, their thinking in the polar form seemed to be less advanced than in algebraic or Cartesian representations: they had a mostly *process* understanding of simplifying a complex fraction using the polar representation, since almost every student was preoccupied with the details of the calculation (compare with box 6.1). From our data we are identifying a *process* understanding as: concerned with the details of the calculation, does not look ahead to see if the calculation is likely to work, does not consider possible "short cuts" in the calculation, and does not bring prior knowledge to bear on the present calculation. In this case, for a student to exhibit *object* understanding of the polar form they would have to show recognition of how closely the polar form is related to geometric methods, in effect some sign of what we have identified as stage 3 and stage 4 thinking. Some students did use a picture to find the angles in questions 6 and 7, but we believe this is still stage 2 thinking in a *process* mode, because they did not use their picture to analyze the problem or otherwise obtain the answer. The purpose of their picture was solely to find the angle using trigonometry. The best evidence of a *process* understanding is that those students who drew a "trigonometric" picture failed to realize that their picture could be used as part of a geometric solution. To these students the trigonometric picture was nothing more than a device to find the arguments, which were needed to do the rest of the computation. Thus, it appears that almost all the students interviewed had a *process* understanding of the polar form, and certainly none had a clear *object* understanding. We have not attempted to further identify action and process (APOS) understandings in stage 2, because our data is somewhat obscured by the difficulties that students had with trigonometry. For example, a student might have a good understanding of the polar form in a conceptual way, but appear to be applying the rules routinely because they are unsure of the trigonometry. Thus, from our data, it is unclear if such a student is exhibiting action or process understanding in the APOS model.

Stage 3

Recall that stage 3 thinking was identified as "the arguments of the numerator and denominator appear to be related (either equal or negatives), so cancellation or polar form should be easier". We have noted that one student did question 3 $\left(\frac{2+2i}{1+i}\right)$ using cancellation, and that two others began their solutions to question 5 $\left(\frac{|a+ib|}{a+ib}\right)$ using a symbolic approach. This shows some ability, even good ability to shift representation wisely, as well as a knowledge of some of the symbolic methods (if we count cancellation as a symbolic method). However, these students were not able to use the symbolic methods (except for cancellation) to finish the questions. Thus, these students appeared to have an awareness of symbolic methods and some knowledge of when to shift, even though they could not actually use symbolic methods.

The only conclusion we think our limited data on stage 3 thinking supports is that the Lest, Post, and Behr criteria (with our addition) do not have any order, in effect, a student might have any level of understanding of any one or more of the criterion. For example, the student who showed the most advanced thinking as far as shifting representations was concerned, was one of the least able to do calculations in a particular representation. This would be an interesting phenomena to study in a further research project. In any event, we have studied the method of cancellation in some detail in a later section in connection with another set of questions.

6.3.3 Summary of Shifting Representations

In summary, we have found that the students we studied were proficient with the Cartesian and polar forms, somewhat less proficient translating from one form to the other, and at the time of the interviews, nearly half the students did not have good judgment about when to shift to a simpler representation. We think our data supports the Lest, Post, and Behr criteria for understanding fractions: 1. Awareness of several representations. 2. Fluency with each representation. 3. Fluent translation ability. In addition, our data suggests that a fourth criteria needs to be added, at least in the case of complex fractions: 4. Good judgment about choosing and shifting representations.

We also believe our data shows that most students had an object understanding of algebraic and Cartesian vector methods, but only a process understanding of the polar vector methods.

6.4 i is a Unit Vector

We asked a number of questions during this study from which it became apparent that many students think of i as a unit vector in the imaginary direction. In this section we analyze the data from two questions that we think supports this conclusion, discuss what is wrong with the unit vector picture, and investigate the origins of the misconception.

From our perspective the problem we are discussing in this section is fundamentally different than the problem of shifting representations covered in the previous section: the question here is 'Can the idea of direction and vectors that is familiar from Calculus on R^2 and Linear Algebra be extended to the Complex plane?'. Put another way, is it appropriate to use the analogy of R^2 to understand the vector representations of complex numbers? This question is far more focused than the question of shifting representations.

Thus our analysis in this section, on i as a unit vector, is from the perspective of analogy, which we describe presently. In an article on the role of algorithms and analogy in the history of mathematics, Phillip Jones [83] describes how

> "Many mathematical steps forward are the result of inductive leaps, spurred by analogy, and facilitated by effective algorithms ..." [p. 13].

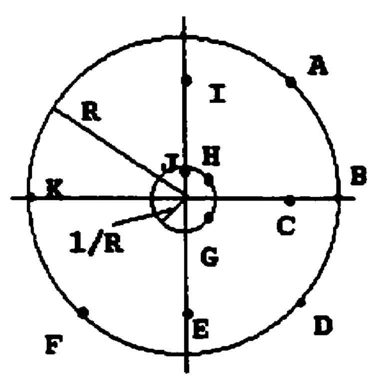
For example, Jones explains how Robert Argand, in 1806, attempted to explain i and -i as mean proportions between 1 and -1, by drawing a unit circle and considering the proportions geometrically. We believe that data presented below can be interpreted as showing that students are readily extending their geometric pictures of R^2 to the complex plane.

6.4.1 Interview Questions and Results

Class 3, Interview #1, Question 6

The first question we will discuss is the most revealing indicator of student thinking of i as a unit vector. This question was class 3, interview #1, question 6 (we have only included the relevent parts):

> 6. Use the diagram to fill in the blanks below. Select a point for each item. The large circle has radius R, and the small circle has radius 1/R. The first one is done for you (and tells you where z is).



i) z is ___A___ v) Im ($\bar{z}$) is ______ x) Im z is ______

ii) |z| is ______ vii) |$\bar{z}$| is ________

The results that interest us, in this section, that came out of this question are that two thirds of the students interviewed said that Im($\bar{z}$) is point E, and that Im z is point I. We believe that the correct answers, within the context of the course are: there is no point that best represents Im ($\bar{z}$), and Im z is best represented by C. The fact that there is no point to best represent Im ($\bar{z}$) was an oversight, but none of the students discovered this. We discuss the correct answers further in section 6.4.2. We have tabulated the complete results for parts ii), v), vii) and x) in table 6.2 .

Table 6.2

Results from Question 6, Interview #1, Class 3

Part	B	C	E	I	R
ii) $\|z\|$	2/6	0	0	0	5/1
v) Im($\bar{z}$)	0	0/2	8/6	0	0
vii) $\|\bar{z}\|$	2/6	0	0	0	5/1
x) Im z	0	0/2	0	8/6	0

The entries in the table before the slashes are the number of students who chose the point heading the column as representing the point described in each part *before rigorous questioning on our part.* During interviews we challenged students quite vigorously on their answers to this question, all but telling them that they were incorrect in some instances. The entries after the slashes indicate the number of students who chose the point in the column as the best representation, *after vigorous questioning on our part.* For |z| and |$\bar{z}$| one student gave the answer $\sqrt{C^2 + E^2}$, which has not been entered in table 6.2. In addition, one student couldn't do the question (recall that a total of 9 students did interview #1, in class 3).

In addition, to the numerical data, we think the interview excerpts given below (taken from student interviews on this question) strongly support the conclusion that many students are treating the real and imaginary parts of complex numbers as vector components.

M1. Ah, for the next one, the imaginary part of z, z is point A, the imaginary point is I.

P. Why is that?

M1. You're simply looking at the vertical component of A.

P. OK.

M1. --, I don't know any better way to explain it.

P. OK, are you saying that the imaginary part is real or imaginary?

M1. ....Yeh, it is a real value, but in the complex direction.

P. OK. The complex direction being pure imaginary?

M1. Yes.

P. So you're saying the imaginary part of z, includes the i then?

M1. So, yeh, it's, the imaginary part of z is, the magnitude of the vertical part, in units of i.

P. Times i?

M1. Times i.

And the other student:

P. Yeh. OK. -- if you are thinking about the imaginary part of z or z bar, whatever it is, as a complex number where would it be?

C. Um, ...It would be E.

P. E?

C. Yes. E for z bar, and I for z.

P. OK, if I take, if z is equal to x plus i y,

C. Yep.

P. What is the imaginary part of z?

C. It's y.

P. It's y. Would that be real or imaginary?

C. It's imaginary.

P. The y is?

C. The y is a real number, but it gets multiplied by i.

P. OK.

C. So it appears here, because this is the i axis.

P. OK. Alright, so let's go on.

These interviews show quite explicitly that these two students are treating i as a unit vector. On the other hand, data from the next question has to be analyzed in some detail before it becomes clear that part of the conceptual difficulty involves thinking of i as a unit vector:

Class 3, Interview #2, Question 2

In class 3, interview #2, question 2, we asked: 2 a) Is $3i < 5i$? We expected some students to give the correct answer to this question, so we planned to ask them if $3 < 5$, if they answered correctly to question 2 a). Our solutions to these questions can be found in appendix 4 and the results are tabulated in table 6.3.

Table 6.3

Results of Question 2, Interview #2, Class 3

Question	Attempted	Yes	No	Yes, Unsure	Analyzed	Restricted
3i < 5i	8	2	1	2	2	2
3 < 5	8	4	1	3	2	2
lexicographic	2	0	1	0	1	0

In table 6.3, the columns headed "Yes" and "No" are the number of students who were sure of these answers. The column headed "Yes, Unsure" indicates the number of students who said yes, but were not sure why. The column headed "Analyzed" is the number of students who eventually concluded (with lots of help from us) that there was no order on the complex numbers (if the usual ordering rules are to hold). Finally, several students found a restricted ordering of some sort, which is recorded in the column headed "Restricted".

We believe the following interview illustrates that the basis for the misunderstanding held by students who answered these questions (one or both) affirmatively, is thinking that i is a unit vector. Notice how K repeatedly refers to the idea of direction, which is equivalent to thinking of the imaginary and real axes as having unit vectors associated with them.

Excerpt of Interview #2, Class 3, K., Question 2

K. Is 3i less than 5i? Um, less than, ..I guess, I'm not sure what they mean. What exactly do they mean, but the modulus of that is certainly less than. It's less than that on the imaginary axis.

P. Well, that is a way you could look at it, but we're not looking at the modulus, I'm just asking you, do you think that's reasonable that three i should be less than 5i?

K. Um, it looks reasonable to me, but there's not really a definition of greater than and less than is in the complex plane.

P. OK, well, let me ask you this. As complex numbers is 3 less than 5?

K. Yes.

P. It is. OK. So let's write that down, $3 < 5$.

K. OK.

P. If I add i to both sides is that OK?

K. I'm not sure, because if you were to multiply that by negative one, then you change the.....as ah, ---what it needs to be when you multiply by i.

P. Well, I just want to add i to both sides.

K. Just add i?

P. So that's still OK?

K. Yeh.

P. So i plus 3 is less than 5 plus i?

K. OK. So I guess in that case you are looking at the modulus.

P. OK, can I add three i to both sides? Instead of i?

K. Oh, ...[laughs], um, go back, I guess if you can add three i, you'd just be adding individual i's., right? I'm not sure you can add i's to both sides. Um, because, there's once there's greater than or less than what do you mean?

P. Well suppose we want it to be an order that obeys the usual rules.

K. OK.

P. You might have to extend it, because we have i now, but we just want all the rules that already hold to hold, we might need some more, but,

K. OK.

P. So that is what we mean by it.

K. OK, but how does less than, how's our, -- all the rules.

P. Well you mean the rules, like, multiply through by a negative, flips the sign, it's got to be transitive, if 3 is less than 5, and 5 is less than 8, then three is less than 8,

K. OK, so if we follow those rules, then I guess it's true. .....Add three i to both sides, you come up with 5 + 4i, and 3 + 4i, and, um,

P. So are you OK with that? Maybe, not too comfortable, but,

K. I guess it's just the --, yeh, I think, this is ----OK.

P. OK. So let's backup to three less than 5, and what happens if you multiply by i?

K. You have negative three less than negative 5, which isn't true.

P. OK, so you're backing right up to three i less than five i?

K. Yes.

P. But you never agreed to that did you?

K. No.

P. You did agree to 3 less than 5?

K. Yeh.

P. So I suggest we multiply that one by i.

K. OK, and that works. But if you multiply this by i then it doesn't work.

P. OK, so?

K. I don't think it's possible to multiply by i.

P. OK. So what are you saying then? We can have this order as long as we don't multiply i?

:
:

K. Yes. Well I think that order is true if you just look at numbers on the real axis. If you just go in one direction, you can say one happens after the other. For the complex plane, if you just go in one direction, right?

P. Hmhm.

K. It doesn't matter if it's real, you can order stuff if it's like this, but you have to specify a direction in which to order stuff. This is only true if you specify an order.

P. And um, but what happened to the multiplication rule? We found that, I mean i is certainly on the imaginary axis. So there's a number on the axis and we can't multiply by that to make it work.

K. OK. But if you followed this in this direction y here, then it would be true.

P. OK.

K. But, I guess if you follow along in this direction it wouldn't be true. It depends on what direction you go, you follow?

P. Yeh, I understand what you mean, and in that sense 3i is less than 5i, right?

K. Yes.

P. Ok, but we don't have a multiplication rule, is that true?

K. Yes, there is no multiplication rule.

P. OK.

K. If it's not, I guess there's still a multiplication rule for real numbers, but it doesn't follow for i,

This interview excerpt shows that K clearly is thinking about the ordering problem in terms of directions, in particular, the imaginary axis is a direction, in effect a unit vector.

6.4.2 Analysis of Question 6 (Class 3, Interview #1)

We begin our analysis of this question by first discussing our solutions to parts v) (Im ($\bar{z}$)) and parts x) (Im (z)). We have already noted that we regard the correct answers to be points -C and C, respectively, on the real axis, and that most students regarded the correct answers to be points E and I, respectively, on the imaginary axis. It has been suggested to us that the problem is somewhat ambiguous, in that it is unclear if we wanted Im(z) interpreted as y or (0, y). In the first place we emphasize that class 3 did not use the vector form, z = (x, y): As stated in the text, "Moreover, in keeping with our philosophy of avoiding pedantics, we have shunned the ordered-pairs interpretation of complex numbers and retained the more intuitive approach (grounded in algebraic field extensions)." Saff and Snider [41, p. ix]. The class 3 instructor consistently avoided the z = (x, y) form. Thus, class 3 students were not faced with the problem of distinguishing between (0, y), y and (y,0), so that even if there is an ambiguity in this question for an experienced mathematician, class 3 students did not have this choice.

On the other hand, if we consider z in the algebraic representation, z = x + iy (which was used as a vector representation in class 3), there is little ground for arguing that there is an ambiguity, since there is no need to distinguish between y and (0, y) in this notation. There is no evidence in our data (that we can find) that any student confused y with iy. Accordingly, we believe that, within the context of class 3, the only correct response for Im (z) is point C.

Having said that what it all boils down to is that when we plot a complex function do we plot the domain or the range? If we do the plot on one set of axes, as is common, then we plot the range, since the domain is usually not interesting. Therefore, plotting y on the imaginary axis is tantamount to plotting the domain of the Im(z), whereas, the default procedure is to plot the range, that is Im(z) should be plotted on the real axis. Thus, this question maybe confounded by the complications of plotting complex functions, but we believe the results are interesting anyway.

As already mentioned, we believe our data on these two questions shows that most students interviewed had a strongly held conception of i as a unit vector (and 1 as a unit vector). The arguments in favour of this conclusion that come from the first question are:

1. All eight students initially chose E and I on the imaginary axis, for parts v) and x) respectively, and six out of the eight stuck to their answers even after close questioning. In addition, some students actually explicitly stated they were treating the Im(z) as having a direction along the y axis (see two interview excerpts below).

2. Even though five students chose R as the answer for parts ii) ($|z|$) and vii) ($|\bar{z}|$), after questioning all but one changed their answers to B. This suggests that students did understand the diagram.

Arguments against the conclusion that the students interviewed essentially think of i as unit vector, are:

1. Students didn't understand the diagram as evidenced by the five students who said the answer to parts ii) and vii) is R.

2. It is unlikely, at this stage in the course, that students have made the conceptual

leap from treating Im(z) as an object (component of z), to thinking of Im(z) as a function mapping C onto R, especially as the diagram is confined to one set of axes. If students were thinking in terms of the former then possibly our results on this question are confounded by the ambiguity mentioned above: students were not distinguishing between y and iy. If students understood Im (z) as a mapping, then we cannot see any explanation for their responses on this question other than that these students were treating i as a unit vector. Thus, depending on which understanding students had of Im(z) it is possible that our diagram does not fairly assess student knowledge.

To decide amongst these arguments, we need to look further. As can be seen from the interviews excerpts of the first question , we pushed students quite hard on the question of where Im(z) should be in the diagram. Nevertheless, only 2 out of the 8 students who said that the answer to Im($\bar{z}$) was E, on the imaginary axis, changed their answers to C, on the real axis.. Since students did very well on other parts of the question, we think our data shows that they did understand the question, but have a solidly held belief that i can be treated as a unit vector (also 1 can be treated as a unit vector).

6.4.3 Analysis of Question 2 (Class 3, Interview #2)

Table 6.3 shows that only one out of eight students definitely rejected the ordering on the complex numbers (using the usual rules) (Recall that question 2 was "Is 3i < 5I ?". Unfortunately, most of the students did not know how to analyze this question. By this we mean that students did not think of listing the rules for manipulating < on the real numbers and attempting to apply (extend) those rules to the complex numbers.

This meant that we had to help them substantially. For example, we had to remind most students about the usual rules, such as, multiplying by a negative number reverses the inequality. In addition, with the exception of the one student who said no to the whole thing, we had to suggest to all the students the idea of trying to define how i fits into the ordering scheme. Three students arrived at partial orderings of some sort after we intervened by showing them the rules for < for the real numbers. All of these students seemed to be well aware that their proposed orderings were of little use (they realized this by themselves), and their persistence with the orderings they found seemed to be due to over enthusiasm, rather than being unaware that the ordering we choose needs to be compatible with the multiplication and addition operations. (For the purpose of solving inequalities, the ordering needs to be compatible with the addition and multiplication operations.)

Another important point of confusion that students had when doing this problem was that at least four students had to work hard to keep from thinking that the modulus function induced an ordering on the complex numbers: these students thought that $3i < 5i$, because $|3i| < |5i|$. The ordering induced by the modulus function is certainly useful geometrically, for example, to define disks, but since it doesn't obey the usual rules (if $|a| < |b|$, it is not true in general that $|a + c| < |b + c|$), it doesn't qualify as an ordering on the complex numbers. We have discussed the ordering in much more detail in section 4.2.5.4.

The results from the ordering question, while somewhat unclear, gives further evidence to support our claim, first made in section 6.3.2 that many students think of the imaginary axis as an actual direction, in the sense of vector calculus. Notice how in the interview, K. concludes that the ordering on any line in the complex plane is preserved if we multiply by a real number. K is clearly trying to extend the idea of a line in R^2 (through the origin) as a subspace (over R), to the complex plane. This works in R^2 because there is no vector product in R^2, so we do not have to worry about something analogous to complex multiplication, but the extension to the complex plane just does not work: the only sub-fields of the complex numbers are on the real axis (real numbers, rational numbers, etc.). Thus, because of complex multiplication, no line except the real axis is a sub-structure (fields, spaces, etc.) of the complex plane.

Thus, on balance, we think the interview evidence (given in section 6.3.2) is compelling enough to conclude that many students are treating the real and imaginary components of complex numbers as vector components, in effect, with an actual direction attached to them, as opposed to scalars. We believe that the origin of this misconception was that, in all three classes studied, the algebraic representation, $z = x + iy$, was represented geometrically in a plane, so students are making the natural conceptual leap of assigning a direction to the x and y axis in the usual vector sense. In effect, i is treated as a unit vector.

The problem of ordering the complex numbers is addressed in the class 3 textbook (page 6, exercise 30), but was not covered in class, according to our notes. As usual in an introductory complex analysis course, the emphasis is on what works, rather than on what doesn't work, so it is not surprising that the fact that the complex numbers cannot be ordered is not emphasized. Nevertheless, it is of some interest that students were not able to successfully analyze this question, without lots of help from us.

The main difficulty with treating i as an unit vector arises in using the extended complex plane and the one point compactification topology that goes with the extended complex plane. These topics were covered in class 1 and class 2, but we have no data on the one point compactification topology, so we do not know to what extent treating i as a unit vector is a problem later in the course.

We would like to return to the issue of direction in the complex plane from a topological point of view (for background to this discussion see section 4.2.5.3). We have already noted in the sections on vector representations of complex numbers that many of the students interviewed have a strongly held vector picture of complex numbers. In particular, the imaginary axis represents an actual direction, in the sense of a real coordinate axis. Geometrically, at least on a local scale, this makes some sense, because we can use the modulus ordering to order the points along θ = constant curves. We have seen, however, that there is no ordering that is compatible with the multiplication and addition operations. So taking into account multiplication and addition, the usual Cartesian representation is quite misleading, because we do not have direction in the complex numbers. In other words, the usual vector representations give a label to each complex number and organizes the complex numbers in a way that gives the addition and multiplication operations a convenient geometrical representation, but that is all: it is an incorrect extrapolation of the vector representation to infer directions in the sense that we have direction in, say, R^2.

The distinction we are making here is not easy for students to understand, but the key to understanding that there is no inherent direction in the complex numbers, as there is in R^2, is understanding that there is no ordering. Accordingly, it is somewhat surprising that so much is made of the vector picture in an introductory course, but that the inevitable extrapolations that students are making are not tempered by a more in depth study of the ordering question. In any case, as long as the extended complex plane is not used too much the misconception is not too serious, since the vector representation is locally adequate.

6.4.4 Summary of i is a Unit Vector

We have found substantial direct and indirect evidence that students view i as a unit vector (also 1 is viewed a unit vector). Most students interviewed appear to firmly believe that the imaginary part of a complex number should be represented on the imaginary axis, rather than the real axis. Our data is not detailed enough to determine if the belief is held due to confusion about plotting the domain or range of Im(z), or is due to a belief that i is a unit vector. In addition, the students interviewed appeared to be extrapolating the notion of direction from R^2 onto the complex plane.

6.5 Basic Facts and Calculations

Under the Shifting Representations section (6.3) of our Multirepresentations theme, we looked at how well students were able to judge when to shift to another representation to simplify a problem. We have also collected a large number of results about the individual Cartesian, polar, and symbolic (or algebraic) representations of complex numbers in this section. We have organized these results into the headings, Cartesian calculations, polar calculations, and symbolic calculations. Thus, in this section we will analyze results of questions that showed students having various problems with each of the representations of complex numbers, studied separately.

6.5.1 Cartesian Calculations

In class 3 we attempted to investigate in detail students' understanding of several aspects of the z = x + iy and z = (x, y) representations. We were interested in how firmly students understood that these representations treat the complex numbers as a vector space, V over a field K, where $V = R^2$, and K = R (in other words, x and y are real). Complex multiplication is then just a vector product that defines a field structure.

In this section we will report on two questions that we asked about Cartesian calculations. The first question was intended to study how well students understood that x and y are real in the z = x + iy and z = (x, y) representations. The second question was intended to study student understanding of simplifying a complex fraction by cancellation of a common factor from the numerator and denominator of the fraction.

6.5.1.1 Cartesian Calculations: x and y are real

In the first of these questions, we asked students about the vector representation, z = (x, y). Unfortunately, (as already mentioned in section 5.4.2) the textbook and instructor for class 3 did not the use the z = (x, y) representation, but the results were useful anyway.

6.5.1.1.1 Interview Questions and Results

This question was asked in the third week of classes, and there were nine respondents.

Class 3, Interview #1, Question 2:

2. If we use the z = (x, y) representation for complex numbers, which of the following are correct statements? (Circle the correct ones)

a. $3 + 6i = (3, 6)$

b. $3 + 6i = 3i(-i, -2)$

c. $3 + 6i = 3i(-i, -2i)$

d. $3 + 6i = (3, 6i)$

e. $3 + 6i = (2 + i, -i + 5)$

f. $3 + 6i = (1+2i)(1, 2)$

g. $3 + 6i = 3(1, 2)$

h. $3 + 6i = (2 + i, 1 + 5i)$

The results are tallied in Table 6.4.

Table 6.4

Results of Question 2, Interview #1, Class 3

Part	**a.**	**b.**	**c.**	**d.**	**e.**	**f.**	**g.**	**h.**
Correct	9	0	7	0	3	0	9	0
Incorrect-Simplified	0	9	0	3	0	7	0	4
Incorrect-Complex	0	0	1	5	3	1	0	3
Incorrect-Unmatched	0	0	0	0	3	1	0	2
Incorrect-Other	0	0	1	1	0	0	0	0

The entries in the row denoted "Correct" show the number of students who said the given part was correct. There were essentially three reasons given by students for rejecting an expression as incorrect: 1) The right hand side of the equation didn't simplify to something equal to the left hand side. 2) The expression was ruled invalid because x or y, or both were imaginary or complex in the given expression. 3) The expression was rejected by inspection because the coordinates obviously weren't equal. The entries in the rows denoted "Incorrect - Simplified", "Incorrect - Complex" and "Incorrect - Unmatched" record the number of students who chose these answers, respectively. The "Incorrect - Other" row is the number of students who rejected the expression for some other reason, such as a miscalculation or they didn't know.

The only parts that are correct are a. and g. The rest are wrong because they involve i's. In the z = (x, y) representation, i is represented by the ordered pair, (0,1), so expressions in this representation that contains i are, by definition of the representation, incorrect. In other words, the symbol i is not used for $\sqrt{-1}$ in the z = (x, y) representation.

The students interviewed demonstrated a wide variety of thinking on these questions: there was considerable variation in the exact sets of correct answers selected by students as well as their justification of these answers. We have included two interview excerpts to show how students thought about some parts of this question.

Interview Excerpt
Class 3, Interview #1, Question 2

K. This one here, ...I think there's something wrong with this [h.] statement where it's got 1 plus 5i, because, for it to be in the imaginary, complex plane, both these numbers have to be real.

P. Hmhm.

K. Because, otherwise it's just, it has no meaning, because this would have, this only has real numbers here, and this only has real numbers here [K. is indicating the x and y components of z = (x, y)], and you have complex numbers in both.

P. OK. So h) is wrong because these [2 + i and 1 + 5i] numbers aren't real?

K. Yes.

P. OK. So what about some of the other ones, they aren't real in this one either [c)].

K. But when you multiply them together they end up being real.

P. Oh, OK, as long as there's a factor of i out there that makes them real?

K. Yes, that's fine.

P. OK. Um, I just, one other thing, um, when you did this one [part f)], I'm not sure if at this point you're rejecting all of these, just because they are not real, but, when you did this one, you rewrote that as 1 plus 2i, and then multiplied, and I'm just curious why you didn't do that with some of the other ones, like you could write this [part h)] as 2 + i, plus one plus 5i all times i.

K. ....Um, I guess because I also think there is only an (x, y) representation on the complex plane, and this one here, um, because the --, the -- get real, in a real system, but this one here is not in the vector form, the (x, y) representation.

P. OK, like I'm just making sure I have this right. You're saying that you can just throw that [h.] out the window because they [the x and y components] are not real.

K. Yes.

P. You just don't even consider it any further, right?

K. Yes.

P. h) we don't consider any further because they aren't real.

K. Yes.

P. But in f) x and y were real, you can put it in the form 1 plus two i? Is that correct?

K. Um, yeh, um, I guess, ..I guess you could put it into this form here, -- algebraically, but this [part h)] wouldn't have any meaning geometrically. Is what I'm trying to say. Whereas this one here [part f)] does have meaning geometrically. You can put this in the form (1, 2) and (1, 2) and both these are real, and you can multiply them together, multiply the modulus and add the angles.

P. OK, so, I just want to get -- on this, so you could put them in here just as an algebraic rule, but what would that mean? Is that what you're saying?

K. Well, I guess you could get some sort of feeling for what that meant geometrically, when you did that, then you could interpret that back into what this function means geometrically.

P. Hmhm.

K. I'll try that.....You still get a wrong answer. Is that right?

P. OK.

K. And now if you have, so you end up with, negative 3, one, two, three, and positive 2. Yeh, I guess, I guess you just have some,

P. OK, so now you're saying this fails because you don't get the right number?

K. Yes.

P. OK. OK, let's go on. We're on to question 3 a) now.

As can be seen, K. vacillates between believing that x and y have to be real, and allowing them not to be real under certain circumstances. The next interview excerpt gives further evidence that students do not understand if the scalars in the vector representation are real or complex (this excerpt refers to part f.) [3 + 6i = (1+2i)(1, 2)]).

M2. ...Ah, same goes in here, ...um , if you take a complex number and multiply, and you multiply by, I suppose it's like multiplying, like multiplying a vector by a scalar, except the scalar is a complex number, so then for f) when you multiply your complex scalar into your vector, which is made-up of real numbers, you end up with complex parts in the --, x and y ---.

P. OK.

M2. -- notation ---.

P. Are you concluding in f), you're rejecting it because you're multiplying it by a complex number?

M2. By a complex number.

P. Not a real number.

M2. Yeh. A real number by a complex number is going to be a complex number.

P. If I rigged these [the question] so that the product was real, would it be OK?

M2. As long as it was (3, 6), yeh.

P. OK,

M2. --, um, yeh, then that would work.

Here we see that M2 does not object to multiplying by a complex scalar, but rather, is arguing that the right hand side of f.) cannot be equal to the left side, because the right side of the equation, after multiplication, will have complex numbers in the two slots of z = (x, y), so this cannot be equal to (3, 6). Thus, we have further evidence that students do not fully understand that x and y have to be real in the Cartesian vector form.

6.5.1.1.2 Analysis of Cartesian Calculations: x and y real

It is possible to look at this question in several ways, for example, as a question primarily about manipulative skills, or a question about the structure of a vector space (we can think of the complex numbers as a vector space consisting of vectors in R^2 over the field R, with a vector product that defines a field on R^2). But we have chosen to focus on the symbolic aspect of the problem. In other words, we are considering this question to be a question primarily of form: did students understand the z = (x, y) representation, in particular, did they understand that the scalars in this representation have to be real? In this form, we expect the numbers appearing in the ordered pair to be real. Any scalars outside the parentheses are real. Thus, without any understanding of the significance of the form, in effect, the vector space structure, students might understand the structure of the form. Finally, when analyzing our data we looked for signs that the students interviewed were obtaining an object understanding of this representation of complex numbers.

Unfortunately, we have found that there is almost no mathematics education literature about the question of mathematical form in the sense we are asking. Most research seems to be concerned with student understanding of the significance of the form in question (for example, limits), or how to improve student thinking by improving their understanding of the significance of the form in question. For example, Dubinsky, *et al* [53], Williams [54], White and Michelmore [55], Thompson [56], and Confrey and Smith [57] are all of these types.

Thus, we have devised our own classification of how students think about form. We have identified four general types of thinking about form from our data on this problem.

1. A mechanical understanding. Students who thought along this line were entirely concerned with whether or not the right hand side of each expression could be translated (mechanically, using whatever other knowledge they could bring to bear on the problem) and simplified to the left hand side. Students who followed this course, were unconcerned about questions of form, or the nature of the $z = (x, y)$ representation. Thus, this type of understanding corresponds to an *action* understanding in the APOS model.

2. Simplification to (3, 6). We believe that students who tried to simplify the right hand side to (3, 6) showed slightly more insight than a mechanical understanding. The reason is that these students appear to be at least using their outside knowledge to bear on the $z = (x, y)$ representation, as opposed to just doing the problem however they

could. Whereas, stage 1 thinking ignores the form of the z = (x, y) representation, stage 2 thinking at least recognizes that there is a new form being introduced in this question, as shown by the attempt to put the answers in the z = (x, y) form. In the APOS model this stage corresponds roughly to the process of *interiorization,* that is, a student in this stage is acquiring a process understanding of the formal structure of the z = (x, y) representation.

3. In stage 3, students recognized that expressions on the right hand side were not correct (except for a. and g.), but attempted to correct them. These students were clearly struggling with what appeared to be incorrect usage of the z = (x, y) representation, but rather than dismiss those expressions in which there are errors, they tried to make sense of them. This shows a more advanced understanding than stage 1 or 2, but also shows that these students do not yet understand that mathematical form cannot simply be altered at will: there are precise rules that need to be followed. This last point indicates that these student do not yet have an *object* understanding of the formalism, but they have acquired a working knowledge, so stage 3 corresponds to a *process* understanding in the APOS model.

4. In stage 4 thinking, students realized the form was not correct, so the expressions were not correct (except for a. and g.). The one student who exhibited stage 4 thinking throughout the question, rejected parts e.), f.), and h.) without any calculation or further consideration, because of the i's appearing in these expressions. Other students showed stage four thinking in their answers to one or two of the problems. The one student who rejected several parts because of improper form likely has an *object*

understanding (in the APOS model) of the formalism of the z = (x, y) representation, whereas the students who rejected one or two parts because of improper form are likely in the process of *encapsulation*.

With these stages of thinking identified we will now give a few examples of how the stages can be used to analyze the results of this question. Rough results for all questions are tabulated in Table 6.4, but to see examples of the four stages of thinking we need to look at interview transcripts and worksheets.

All nine students first determined that (b) [3 + 6i = 3i(-i, -2)] was incorrect by simplifying the right hand side to get various answers not equal to 3 + 6i. None of the students objected to the i's on the right hand side of this equation when they first attempted it (one student recovered later). Thus, all but one student thought about this part of the question as a problem of simplification, with apparently almost no understanding of the structure of the z = (x, y) form. Seven of the nine students interviewed did simplify to z = (x, y) form [(3, -6i) was typical], however, thereby exhibiting what we have called stage 2 thinking.

It is not too surprising that the complexity of the question affected the level of thinking according to our scheme: seven or eight students used stage 2 or 3 thinking for parts (a), (b), (c), and (d), but three students used stage 1 thinking for parts (e) and (f). Presumably this is probably best explained as reductionism: faced with complexities in unfamiliar notation, some students returned to the notation they knew best.

Another variation of what we regard as stage 2 thinking can be seen in the second interview excerpt. Notice that M2. does not object to multiplying by a "complex scalar" if the answer comes out to (3, 6). Of course, in the vector representation of complex numbers we don't have "complex scalars", since complex numbers are vectors in this representation.

A good example of stage 3 thinking appears in the first interview excerpts given above (with K). We can see from the excerpt that K switches back and forth, and is concerned with having a geometric interpretation of the misuse of the $z = (x, y)$ form. Nevertheless, K does not quite reject the misuse of the $z = (x, y)$ form altogether. Notice how K tries to find some reasonable interpretation of the representation with x and y complex, and also tries to interpret multiplication by a complex scalar.

Another example of what might be stage 3 thinking is the case of part e.) [$3 + 6i = (2 + i, -i + 5)$], 3 students decided that e.) was incorrect, because there was no way that $2 + i$ could be equal to 3, and left it at that. These students did not reject $2 + i$ outright, because of the i, as stage 4 thinking would require, but appeared to have changed their thinking from previous parts, where they attempted to simplify the right hand side. Our data does not indicate exactly why students chose a comparison approach on this problem, when they simplified in the previous questions. It might be because they simply did not look deeply enough to find a way to simplify the right hand side to $3 + 6i$. (We expected some students to say the right hand side was equal to $2 + i + i(-i) + 5i = 3 + 6i$, and four students did do this).

Aside from the one student who did the whole question using stage four thinking, some evidence of stage four thinking can be seen in answers to part d.) [$3 + 6i = (3, 6i)$]. All of the students who rejected d.) by inspection, did so specifically because of the i in 6i, and not because 6i is not equal to 6, so this is evidence of stage four thinking. However, we have to be careful, because of the problem we have already noted: maybe these students just had not thought of some way to simplify (3, 6i). Indeed, as we have seen in part e) [3 + 6i = (2 + i, -i + 5)], only one student rejected this part because of the i's on the right hand side. The rest of the students either said this part was correct, or rejected it because 2 + i is not equal to 3. Thus, it is not clear if the answers to part d) show stage four thinking.

It is interesting to look a little further on parts d) and e), because seven out of nine students changed their thinking between parts d) and e). Six students who rejected part d) because of the i in 6i, said e) was incorrect because 2 + i was not equal to 3, and there was no way to make it equal (say by multiplying by a scalar). One student said part d) was wrong by calculation, but said part e) was wrong because of the i in 2 + i. We think that the six students who went from stage four thinking to stage two or one between parts d) and e), probably did so because of the new complications introduced in part e): x and y are complex numbers in this part, as opposed to real or pure imaginary. In addition, there is no scalar factor. Apparently there is a clear conceptual leap between pure imaginary components and complex components even at third year university level.

We think this question could be modified in several ways to gain considerably more understanding into how students are thinking about these questions. For example, the order of the questions could be changed, or say, a question like part e) with a scalar factor could be added to the list.

In addition, our analysis has ignored the question of whether or not this question is confounded by the problems students have with the equal sign. For example, students sometimes treat "=" as though it is the word "is", or treat "=" as though it was a vertical line between debit and credit ledgers in bookkeeping. We have ignored this question because we can see no evidence in our data that students did not correctly understand the equal sign in each question. That is, it appears that all nine of the students interviewed understood that the task was to see if the two sides of the equation were exactly the same complex number. There were of course, various levels of success of integrating this understanding (of the equal sign) with other aspects of the questions, such as, how to handle the i's. Nevertheless, it would be interesting (and important) to investigate the question of the equal sign further (or possibly revise the question so as to exclude the equal sign).

6.5.1.1.3 Conclusion of Cartesian Calculations: x and y Real

In conclusion, we have analyzed our data on Cartesian calculations - x and y real as a mathematical form. We have identified four stages of thinking about this particular form that may well be generalizable to all mathematical form. In any case, we think this particular question has opened a door to questions of mathematical form that warrants further research.

6.5.1.2 Cartesian Calculations: Cancellation

Under Cartesian calculations in our Basic Calculations category, the second type of question we asked that shed light on students' understanding of the $z = x + iy$ representation involved the possibility of canceling a common factor from the numerator and denominator of a rational expression. Hence, we have called this section Cartesian Calculations: Cancellation.

6.5.1.2.1 Interview Questions and Results

The questions we asked here were as follows:

Class 1, Interview #1, Part 3, Question 3 and Class 2, Interview #1, Part 2, Question 3:

3. $\dfrac{2+2i}{1+i}$

Class 3, Interview #1, Question 3:

3. a) Put the following into a + bi form.

i) $\dfrac{2+4i}{1+2i}$ ii) $\dfrac{-6+3i}{1+2i}$

b.) Three students did the following problem three different ways. Which of them are correct? (Circle the correct ones)

i) $\dfrac{6-9i}{2-3i}=\dfrac{6-9i}{2-3i}\cdot\dfrac{2+3i}{2+3i}=\dfrac{12+27}{4+9}=\dfrac{39}{13}=3$

ii) $\dfrac{6-9i}{2-3i}=\dfrac{6-9i}{2-3i}\cdot\dfrac{2+3i}{2+3i}=\dfrac{12-11i+9i+27}{4+9}=\dfrac{39-2i}{13}$

iii) $\dfrac{6-9i}{2-3i}=\dfrac{3(2-3i)}{2-3i}=3$

c. Put the following into a + bi form.

i) $\dfrac{-4+2i}{2-i}$ ii) $\dfrac{6+2i}{-3-i}$

In these questions we were interested in whether or not students would realize that they could factor out a real number from the numerator, and then cancel a common complex factor from the numerator and denominator. For example, we wanted to see if students would simplify $\frac{2+2i}{1+i}$ as follows: $\frac{2+2i}{1+i} = \frac{2(1+i)}{1+i} = 2$. Put another way, we were interested in determining if students realized that $\frac{a+ib}{a+ib} = 1$.

In addition, this question seemed to bring forward student confusion about basic facts. To see this we have two interview excerpts taken from the discussion of question 3 b iii), Interview Number One, Class 3: $\frac{6-9i}{2-3i} = \frac{3(2-3i)}{2-3i} = 3$. In the first excerpt, M1 is uncertain whether they can cancel a complex factor from the numerator and denominator of a complex fraction. In the second, C is not sure what the scalars are in the $z = x + iy$ form of complex numbers.

Interview Excerpt Number One
Question 3 b iii), Interview Number One, Class 3

M1. Alright, so part 3? .....OK, same quotient of complex numbers, ...so I look at the numerator, and they have factored the numerator, they've taken a 3 out of it,

P. Hmhm.

M1. And they've done that correctly. So I see that, this 2 minus 3i cancels in the numerator and the denominator, to get the right answer of three. So part three is correct as well.

P. OK, summarizing, in 3?

M1. Summarizing in three, it was not necessary to multiply by the complex conjugate of the numerator, because we could factor the numerator, into some, and it canceled the denominator.

P. And that's, that's allowed? To cancel a complex number from top and bottom?

M1 ....Um, ...yes. A complex number divided by itself is one.

P. OK, are you sure?

M1 No, I'd have to work out a general form a + ib, and then --.

P. OK.

M1 Intuitively, I'd say yes,

P. OK, so you,

M1 I would need to, I would need to check that, in a general form.

P. OK, you can check it on the side there if you want.

M1. So I would say that part three is correct. You can cancel the complex numbers.

During this section of the interview, M1 attempted to supply a proof of the cancellation rule. Although they (we have used "they" to refer to M1 and other students being interviewed, as a confidentiality precaution to avoid identifying gender) thought they were successful, they were, in fact, incorrect (see below). Thus, it is apparent from this excerpt that M1 is not entirely sure of the basic rules of Cartesian calculations (cancellation, in particular). Having verified this rule (for themselves, in any case), M1's confidence improved, as shown by the last line. In the next interview excerpt we have more evidence that students are not sure what the scalars are in the z = x + iy formulation of complex numbers.

Interview Excerpt Number Two
Question 3 b iii), Interview Number One, Class 3

C. And this is not correct. Oh, they didn't, they um, they factored. and --, this is also correct.

P. OK. So in this one, what did they do again?

C. They factored the numerator, they just took the scalar value of 3, and then multiplied, they're left with 2 minus 3 i, if you multiply through you get 6 minus 9 i, and then you can just cancel these [(2-3i)] out, and you get the answer.

P. OK. Can I just ask you, you referred to 3 as a scalar, can you elaborate on what you mean by that?

C. Oh, ah, this is just a real number, You can just multiply, 6 and 9 have a common factor of three,

P. Hmhm.

C. And then you pull that out of the complex, the rectangular expression for, -- a lot like vectors, so this like multiplying a vector by three.

P. OK.

C. And that's legal as far as I know. [laughs]

P. Can you tell me what the allowed scalars are? Are they just real numbers or complex numbers?

C. No, [This 'no' refers to the part of P's question: "Are they just real numbers?"] you should be able to factor, you should be able to factor even complex numbers. ..both the imaginary and the real component.

P. OK. So, if I found some way to factor this or another problem as two complex numbers, and then cancel that would be OK?

C. Yes, --.

From this excerpt we can see that C includes complex numbers amongst the scalars in the $z = x + iy$ form.

6.5.1.2.2 Analysis of Cartesian Calculations: Cancellation

The results for question 2, classes 1 and 2, and question 3, class 3 are summarized in table 6.5.

Table 6.5

Number of Students who Used Cancellation to Simplify Various Questions

Question Number	Yes/Total
Number 3, Class 1	2/5
Number 2, Class 2	0/6
Number 3 a i), Class 3	1/9
Number 3 a ii), Class 3	0/9
Number 3 c i), Class 3	3/9
Number 3 c ii), Class 3	5/9
Cancellation Method is Correct	6/6

The entries are the number of students who simplified by cancellation (as opposed to realizing the denominator), followed, after the slash, by the total who attempted the question. The results of question 3 b) was the same for all nine students asked this question: i) and iii) are correct, and ii) is incorrect.

According to M1's worksheet (and the section of the interview omitted above) he "proved" the cancellation rule for complex quotients as follows:

$$\frac{a+ib}{a+ib}=\frac{a+ib}{a+ib}\cdot\frac{a-ib}{a-ib}=\frac{a^2+b^2}{a^2+b^2}=1\ .$$

This is not a valid proof since we need to know that $\frac{a-ib}{a-ib}=1$, which is what we are trying to prove. Of course, at third year university level, proofs of basic algebraic facts, such as, $\frac{a+ib}{a+ib}=1$ can be quite challenging, so we cannot take M1's inability to supply a proof as strong evidence that M1 did not understand the cancellation rule. The main evidence that M1 is not sure is that they said they were not sure during the interview.

In any case, it would be interesting to investigate how students would supply a proof of $\frac{a+ib}{a+ib} = 1$ [To prove this we need the definition of division, which says that $\frac{a+ib}{a+ib} = 1$, if and only if, a+ib = 1(a+ib). Then we just have to use the definition of complex multiplication to show that the multiplicative identity for the complex numbers is the real number 1, so that 1(a+ib) is just a+ib].

The results show that if students are alerted to the possibility of cancellation they will use it. This can be seen in the increased use of this method between parts a) and c). One student commented that cancellation did not seem very likely in most cases, so that they would not normally look for it. However, this student decided that cancellation was a good idea when it was available.

As might be expected, we found that most students revealed that they had various misunderstandings as they worked through these problems. For example, C's interview is notable because it gives some indication of how student confusion about vectors and scalars in the vector representation appears in a practical problem. C's contention that we can cancel a common complex factor even if there are other complex factors in the numerator or denominator is certainly correct. But it appears that C. does not distinguish between "complex scalars" and real scalars, as evidence by the response to the question: "Can you tell me what the allowed scalars are?"

In addition to these two samples, other students speculated that the cancellation method would save time and be more accurate. Furthermore, of the six students (in class 3) that we questioned closely on the validity of the Cancellation Method, all six were confident that the method was correct, although just two students attempted to prove it for themselves.

Thus, it appears that students are not learning the cancellation rule (recall that almost no one in classes one and two attempted $\frac{2+2i}{1+i}$ by cancellation). Once they are instructed, they find it useful and time saving. We believe that there is not too much of a conceptual problem here: students simply are not learning that they can cancel and mostly do not think of it themselves. Of course, teaching them might involve more than simply telling them: one student emphatically declared that they would definitely have done question 3 b iii) by cancellation, but then did both 3 c i) and ii) by realizing the denominator!

6.5.2 Polar Calculations

Thus far in our Basic Calculations sub-theme, we have primarily studied Cartesian vector forms. In this section we look at some of the data we collected on the Polar Vector Representation of complex numbers.

The polar representation of complex numbers, $z = re^{i\theta}$, where $r = |z|$ is the modulus of z, and $\theta = \arg z$, is one of the principal representations of complex numbers. Recall from section 4.2.3 that the polar representation has several advantages: 1. Multiplication and division are simplest in polar form. 2. Geometric ideas are often easily represented in polar form, for example, the polar parameterization is easiest on many useful contours. 3.

The inherent multivaluedness of the complex numbers is readily (and simply) represented in the polar form.

Even though the polar representation is a vector representation, we have seen in section 4.2.3 that there are several characteristics of the polar form that are not present in the Cartesian forms, such as, trigonometry, polar coordinates, a new multiplication rule, and exponents, so that when we were designing our study, separate questions specifically on the polar form, appeared to be well founded.

Most of the questions we asked students that involved the polar representation were intended to study other topics, such as continuity and analyticity, or shifting between Cartesian vectors and polar vectors (see the section on shifting representations). Nevertheless, we did ask three sets of questions intended to directly study the understanding that students had of the polar representation.

6.5.2.1 Polar Calculations: Interview Questions and Results

The first set of questions comes from the first interview with class 3 students. We asked three questions to test their ability to do basic computations in polar form:

Class 3, Interview #1 (third week of classes):

Question 4 b): Simplify the following given that $z = re^{i\theta}$.

i) $3e^{i\pi} \cdot 5e^{i\frac{\pi}{2}}$ ii) $3e^{i\pi} \div 5e^{i\frac{\pi}{2}}$ iii) $3e^{i\pi} + 5e^{i\frac{\pi}{2}}$

Our second set of questions has already been presented in our Shifting Representations section. (Recall that we asked class 1 and class 2 students a series of questions designed to see how complicated a question had to be before they would switch from using Cartesian methods to polar methods.) The issue of shifting is discussed in that section. It is interesting, however, to study what problems students had with the polar form in those cases where they used it. Questions 6, 7, 8 from interview #1 for class 1 and class 2 students were as follows:

6. $\dfrac{-2+2i}{(1+i)^3}$; 7. $-\dfrac{8}{(\sqrt{3}+i)^6}$; 8. $\dfrac{2(1+i)^4}{(-2\sin(\frac{\pi}{12})+2i\cos(\frac{\pi}{12}))^5}$

There was a slight difference in question 8 between the two classes: in class 1 the argument in the denominator was written as 15° (15 degrees), rather than $\frac{\pi}{12}$. In addition, one student in class one got a different version of interview #1 that didn't include these questions 7 and 8.

The third question we asked that was specifically designed to investigate student ability with the polar form, was about DeMoivre's theorem. When tutoring students we noticed that they sometimes assumed that DeMoivre's theorem has variations. The correct statement is: $(\cos\theta \pm i\sin\theta)^n = \cos(n\theta) \pm i\sin(n\theta)$. To examine this question formally, we asked class 2 students the following question in interview #3:

Class 2, Interview #3, Part 3, Question 5:

5. Find the real part of $\dfrac{(i\cos\theta+\sin\theta)^5}{(\cos 2\theta+i\sin 2\theta)^2}$.

6.5.2.2 Polar Calculations: Analysis

The overall picture that emerges from studying the results of these three sets of questions is that the argument is the problem. We believe that prior to studying complex analysis, most students have been able to ignore questions of multivaluedness. No doubt, many student view the problems presented by multivaluedness of the polar form as a nuisance, and simply try to disregard them, rather than embracing multivaluedness as an essential (and very interesting) aspect of the theory. In any case, we have chosen to analyze this question in terms of the connections between the errors students made and their understanding of multivaluedness of the polar form.

The correct answers to all three sets of questions can be found in appendix 4. In the first set of questions, four students tried to put the answer in polar form in part iii): These students found the correct answer (-3 +5i) and then tried to convert

$-3 + 5i = \sqrt{34}e^{i[\mathrm{Arctan}(\frac{-5}{3})+\pi]}$. All four students got the modulus correct but made errors in the argument. For example, they used arctan x instead of Arctan x, dropped the minus sign in front of -5/3, or dropped the phase angle of π. These kinds of mistakes in themselves are not enough to draw conclusions of a general nature, but they are part of an overall picture. For example, the difference between arctan x and Arctan x is very significant, but complex analysis is perhaps the first undergraduate course a student would take in which failure to understand the difference between the two would lead to significant errors.

The results from our second set of questions is summarized in Table 6.6 (the correct answers are in appendix 4).

Table 6.6

Results from Questions 6, 7, 8, Interview #1, Classes 1 and 2

Question	Class 1 Attempts	Class 1 Correct	Class 2 Attempts	Class 2 Correct
6	4	3	3	2
7	3	1	3	2
8	3	0	5	1

As we have already noted, almost all of the difficulty with these questions was computing the arguments. One student acknowledged that they did not know how to find the arguments, apparently not realizing that they could use basic trigonometry (or perhaps they had forgotten basic trigonometry).

Of the three incorrect or doubtful answers to question 7 using polar methods, two were errors in the modulus and there was one incorrect argument calculation. Nevertheless, the primary difficulty students had (looking at all the student responses) was finding the argument. Two used their calculators.

Finally, the results for question 8 were certainly the worse, with only one correct answer out of 8 attempts. Although some of the errors were arithmetic in nature, several fit into our theme for this section. For example, one student was completely confused by 15°: it did not occur to them to convert this into radians.

Two students found question 8 to be just too complicated (because of the arguments) and gave up. Another student applied DeMoivre's theorem to the denominator (it is not correct to do this directly, but in this case it happens to work), without justification, but could not make further progress. This student explained that they were trying to convert sine into cosine and vice versa using the addition of angles identities for sine and cosine (so they could apply Euler's theorem). It clearly did not occur to this student to factor out i, to achieve the same objective

Thus, there were several types of errors and difficulties on this question: students could not find the argument or modulus correctly for one or more expressions, misused DeMoivre's theorem, did not realize that they could factor out i in the denominator of question 8, and were intimidated by a question that looks hard, but is straight forward if one breaks it into parts. The main difficulty students appeared to have when finding the argument was not being comfortable or fluent with the arctan x function. In particular, they were not fluent with the multivalued qualities of this function.

In our last question of this section, (Class 2, Interview #3, Part 3, Question 5):we looked at DeMoivre's theorem. We can use DeMoivre's theorem to solve this question if we apply it carefully:

$$\frac{(i\cos\theta+\sin\theta)^5}{(\cos 2\theta+i\sin 2\theta)^2}=\frac{i^5(\cos\theta-i\sin\theta)^5}{(\cos\theta+i\sin\theta)^4}=i(\cos\theta-i\sin\theta)^9=i\cos 9\theta+\sin 9\theta\ .$$

Thus, the real part of the original expression is $\sin 9\theta$.

Of the five students who attempted this question, only two attempted to use DeMoivre's theorem, one correctly and the other mechanically. This last student said $(i\cos\theta + \sin\theta)^5 = \sin 5\theta + i\cos 5\theta$, applying DeMoivre's theorem incorrectly, but, nevertheless, getting a correct equation. Evidently, $(i\cos\theta + \sin\theta)^n = \sin(n\theta) + i\cos(n\theta)$, provided $n \equiv 1 \bmod 4$. (When constructing the question we miscalculated, believing that for this particular expression DeMoivre's theorem worked for $n \equiv 0 \bmod 4$). It is clear from this student's interview transcripts and worksheet that they applied DeMoivre's theorem without hesitation. They first attempted to find a factor k, such that, $k(\cos\theta + i\sin\theta) = (i\cos\theta + \sin\theta)$, so they could apply Euler's theorem. It was only when this attempt was unsuccessful, that they applied DeMoivre's theorem.

There is not enough data to make too many conclusions about DeMoivre's theorem, and this is probably a small thing to clarify in any case. Nevertheless, we did observe students misusing DeMoivre's theorem several times during our study. Presumably, students believe that because of the close relationship between $\cos\theta$ and $\sin\theta$, especially in complex analysis, that DeMoivre's theorem works with $\sin\theta$ and $\cos\theta$ interchanged.

6.5.2.3 Polar Calculations: Summary

In conclusion, the main difficulties that students had with the polar representation were lack of practice with basic trigonometry, and particularly the arctan x function, little or no grasp of the multivaluedness of the arctan function, and a tendency to assume that $\cos\theta$ and $\sin\theta$ can be interchanged (since they are so closely related). Although some students had trouble calculating the modulus correctly, they made mostly arithmetic mistakes, so there did not appear be any conceptual problems with the modulus.

6.5.3 Symbolic Calculations

This section contains analysis of data we collected about how well students were able to do calculations involving z, using just the properties of $|z|$, $\bar{z}$, Re(z), Im(z), and the field properties of the complex numbers. The questions we are discussing in this section can be done very readily without substituting $z = x + iy$, or $z = re^{i\theta}$, or without using a geometric approach. Obtaining some skill at symbolic methods is very useful (but not essential) for more advanced subjects, such as, power series and contour integrals. The main result that our data makes clear is that students are not obtaining any fluency in symbolic methods. We have done very little analysis, other than to show how our data supports the theme that students are not obtaining symbolic skill. The reason is that in most cases students did not attempt a symbolic solution.

6.5.3.1 Symbolic Calculations: Interview Questions and Results

We asked a number of direct questions, and questions which had some other primary focus, to look for evidence that students were obtaining symbolic skill. We have summarized the results in Table 6.7, given below. The questions are as follows (questions with answers are in appendix 4):

Class 2, Interview #1, Part 2, Question 11: Show that if $|z| = 1$, then

$$|1 - z\overline{w}| = |z - w|.$$

Class 2, Interview #2, Part 3, Question 1: Find all solutions to $|z - i| = |z + 1|$.

Class 2, Interview #3, Part 3, Question 10: If $|z| = 2$, show that $|2 - \overline{zw}| = |zw - 2|$, for all w.

Class 2, Interview #5, Part 3 Question 3: Show that for $\gamma(t) = e^{it}$, $0 \le t \le 2\pi$,

$$\int_\gamma \left(\frac{1}{z} - \bar{z}\right) dz \text{ is zero. Is } \frac{1}{z} - \bar{z} \text{ analytic?}$$

Class 2, Interview #5, Part 3, Question 8: If γ is a circle of radius 2 centered at i, what is $\int_\gamma \frac{|z+i|}{|\bar{z}-i|} dz$?

Class 3, Interview #1, Question 7 a): Find all solutions of the following equations: i) $z^2 + 2iz - 1 = 0$ ii) $z^2 + (1+i)z + i = 0$

Class 3, Interview #2, Question 4:

a) For which z is $|\bar{z}| = |z|$? Explain.
b) For which z is $|\bar{z}+1| = |z+1|$? Explain.
c) For which z is $|\bar{z}-i| = |z+i|$?
d) For which z is $|iz+1| = |\bar{z}+i|$
e) Where is $f(z) = \frac{|\bar{z}-1|}{|2z-2|}$ analytic?

Table 6.7

Results of Questions Asked to Test Knowledge of Symbolic Methods in Classes 2 & 3

Question	Attempted	Symbolic Correct	Symbolic Attempt	Other
$\lvert 1 - z\bar{w}\rvert = \lvert z - w\rvert$	4	0	1	0
$\lvert z - i\rvert = \lvert z + 1\rvert$	5	0	1	3
$\lvert 2 - \overline{zw}\rvert = \lvert zw - 2\rvert$	1	0	0	1
$\int_\gamma \left(\frac{1}{z} - \bar{z}\right) dz$	4	0	0	4
$\int_\gamma \frac{\lvert z+i\rvert}{\lvert \bar{z}-i\rvert} dz$	4	0	0	0
$z^2 + 2iz - 1 = 0$	7	6	0	1
$z^2 + (1+i)z + i = 0$	4	2	2	0
$\lvert\bar{z}\rvert = \lvert z\rvert$	6	1	0	5
$\lvert\bar{z}+1\rvert = \lvert z+1\rvert$	5	1	1	3
$\lvert\bar{z}-i\rvert = \lvert z+i\rvert$	4	1	0	3
$\lvert iz+1\rvert = \lvert\bar{z}+i\rvert$	3	0	1	2
$f(z) = \frac{\lvert\bar{z}-1\rvert}{\lvert 2z-2\rvert}$	1	0	0	0

Since there are so many questions, we have used the essence of each question to identify it in the table, rather than identify them by class and interview number. The column headed "Symbolic Correct" is the total number of correct answers using symbolic methods. The column headed "Symbolic Attempt" is the number of students who attempted to use symbolic methods, but gave-up altogether or tried another method.

The column headed "Other" is the number of students who got the question correct using some other method. The other methods were usually substitution of Cartesian or polar vector form, or occasionally geometric methods.

6.5.3.2 Symbolic Calculations: Analysis

Examination of table 6.7 shows very strongly that students are acquiring very little proficiency in symbolic techniques. We believe that the results show that students do not have symbolic methods in their repertoire, as opposed to the possibility that they simply prefer other methods, because so many of the questions we asked are dramatically simpler to do using symbolic methods: for example, to show that $f(z) = \frac{|\bar{z}-1|}{|2z-2|}$ is analytic everywhere except at z = 1, using the z = x +iy representation, requires calculation of the modulus of the numerator and denominator, and then further simplification. (Using symbolic methods, we can see, almost by inspection, that f(z) = 1/2 if z is not equal to one.) In addition, sometimes students would ask if there was an easier way to do the question, and be very impressed by the symbolic method. In other words, we believe that for the questions discussed in this section, most students at third year level would prefer to use symbolic methods if they were able to use these methods.

The only exception occurred when they were asked to solve quadratic equations. Quadratic equations are not an especially good indicator, because of the problem of "thinking real, doing complex", it is unclear if students are thinking in terms of symbolic methods at all. If they are thinking of z as just the x of real polynomial theory, then it is not particularly encouraging to see them readily apply the quadratic formula. During interviews we didn't explore this point, so it is unclear if the results of questions about quadratic equations indicate fluency in symbolic methods.

Since students do not seem to be learning symbolic methods, for example, only one student out of six tried to use symbolic methods to solve an equation as elementary as $|\bar{z}|=|z|$, we have examined the textbooks and lecture notes for the three classes studied to see how much instruction students are receiving. The question is: is this topic very difficult, so that even very good students are not getting it, or is there too little instruction and emphasis? This is an important question, because we tutored one student who was not aware that symbolic methods existed.

In class 1 the textbook (Priestley [68]) has two sections, 1.3 and 1.4, on pages 3 and 4, that contain instruction on symbolic methods. Section 1.3 describes properties of complex conjugates, and section 1.4 uses symbolic methods to prove three inequalities. There are no other direct examples, and just two problems, numbers 5 and 6 on pages 11 and 12. These are complicated problems that were assigned for homework, but which can be done substituting one of the vector forms. (We have not included the problems here, because the point of our discussion is that there is *not enough* coverage of symbolic methods.) Priestley is quite terse, and perhaps too difficult for an introductory course at the 300 level, but in contrast to the symbolic treatment there are several pages of discussion of elementary geometric results, such as, lines, circles, and inequalities.

The instructor in class one proposed to prove that $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$, and asked for help from the class. Substituting $z = x + iy$ was suggested, and after commenting that a polar substitution would be better for multiplication, the instructor proceeded with the proof using the Cartesian substitution. To prove the inequalities (triangle inequality) in section 1.4 of Priestley, the instructor of class one, used geometric arguments. While students were assigned the two problems mentioned above (that can be done most conveniently with the help of symbolic methods) for homework, we can find no examples of symbolic calculations done in class.

In class two, sections 2, 3, 4 of the textbook (Churchill and Brown [69]) cover the basic facts about complex numbers under the headings, "Algebraic Properties", "Geometric Interpretation", and "Triangle Inequality". In any case, sections 2, 3, and 4 of this textbook establish many useful facts that can be used for symbolic manipulations. Some of the facts that are established in this section of Churchill and Brown, can be shown using symbolic methods. Although Churchill and Brown do not actually prove any of the results using symbolic methods, sketches are given that show how to prove several facts, with details left to the exercises. For example, on page 5, Churchill and Brown suggest that we can use $\frac{1}{z} = z^{-1}$ to show that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$. Some of the details are the subject of exercise 11, on page 6. After establishing all these results (either directly or with the help of exercises) there are a number of applications or examples of using symbolic methods to solve problems. For example consider, exercise 16, on page 6, and exercises 4, 7-13, 15, 16 page 11.

Class 3 students got slightly more instruction in geometric and symbolic techniques. The instructor (and textbook Saff and Snider[65]) established basic facts using Cartesian substitution, but the class 3 instructor did two elementary proofs using symbolic methods: i) $\frac{1}{\bar{z}} = \frac{z}{|z|^2}$, and ii) $|zw| = |z|\,|w|$. Saff and Snider has four exercises which can be done with symbolic methods: 13., 14., 15., and 16. on pages 11 - 12. In addition, some of the parts of exercise 7, page 11 can be done symbolically (in examples, Saff and Snider use a geometric approach to do the type of question that appears in exercise 7, so it is unlikely that many students would use symbolic methods on exercise 7).

6.5.3.3 Symbolic Calculations: Conclusion

In conclusion, students in all three classes received some instruction on symbolic methods, but not nearly enough for these methods to be accessible to them. This is a serious short coming of the courses we studied, since symbolic methods are very useful, and we believe that practice with symbolic methods would improve students' skills with the other methods, such as the Cartesian or polar methods studied in this thesis.

6.5.4 Algebraic Methods

We asked students in our study several questions about the complex numbers from an algebraic, but theoretical point of view. We were interested in how well students understood concepts such as: the complex numbers as an algebraically complete field, the complex numbers as an algebraic extension of the real numbers, the connection between complex numbers and irreducible polynomials with real coefficients of degree two or less, and finally, the a + ib representation of complex numbers as an algebraic extension as opposed to a vector representation.

We did not really expect students to gain much understanding of these concepts in a first course in complex analysis. We thought they might acquire a working knowledge of some of the ideas. For example, we thought it would be reasonable for students to understand and use the fact that every polynomial of degree n over the complex field has n roots.

Unfortunately, our data on these subjects is very limited, because there was only very limited coverage of these topics in class. In addition, two of the textbooks covered these aspects of complex numbers very briefly, so it became simply unreasonable for us to expect students to have much knowledge of the algebraic structure of the complex numbers. Thus, we have omitted any analysis on these subjects and will confine our comments to a few conclusions.

In conclusion, we found that students received very little instruction on the idea of complex numbers as a field extension in classes one and two, but that there was more coverage in class 3. A similar assessment holds for polynomial theory. We noticed that two students were unable to find the roots of a polynomial using the quadratic formula, because they did not know what to do with a discriminant of -2i. Student responses to interview questions indicate they are absorbing approximately what they are taught.

6.6 Uneven Skill with Basic Material Affects Later Work

One of our main objectives in this study has been to try to identify ways that confusion about basic material affects students' ability to understand later work. We have found many examples of students struggling with advanced material, because they could not do basic manipulations. In most cases, what we are referring to as "basic" in this section is skill with the representations of complex numbers that we have been discussing throughout this dissertation. Thus, for example, how does student confusion about the Re f(z) cause problems for them when they attempt later topics in the course, such as integration or using the Cauchy-Riemann conditions. The examples presented here are a sample from a variety of topics.

Before analyzing our data for this section we note that many other researchers have examined the issue of insufficient preparation affecting later work. For example: Thompson [56] shows how even very mathematically sophisticated students encountered severe difficulties with basic calculus questions, because the mental images they had (if any) of the fundamental theorem of calculus were not rich enough to help them with applications of this important theorem.

Zazkis and Dubinsky [59] have studied in great detail the problems that students had computing permutations of the dihedral group. The particular problem they studied could be directly traced to these students failing to recognize that a permutation has two distinct interpretations which must be used consistently. Thus, a relatively small oversight led to perplexing (for the students) errors later on in the course.

White and Mitchelmore [55] study the difficulty that students have doing basic word problems in calculus. The hypothesis that these researchers tested was whether or not further instruction in calculus concepts would improve student performance. In other words, how was basic preparation affecting student ability to do word problems. They found that most errors could be understood as students manipulating symbols as opposed to quantities to be related. Thus, in terms of the present discussion, student understanding of notation and symbols had not advanced to the object understanding required to build the equations needed to solve word problems.

To analyze the data for this section we have altered our format of presenting all the questions first and then analyzing them, because the questions in this section are unrelated, save for being examples of students struggling with more advanced material because of insufficient skill with basic material. Thus, the common thread between the questions in this section is that they are examples of students struggling with advanced material because of their difficulties with basic material.

6.6.1 Class 2, Interview #2, Question 2

Class 2, Interview #2, Question 2 was: Suppose $f(z) = (2z - x)^2$, where x is the real part of z. Where is f(z) differentiable?

The following interview excerpt shows how much trouble one student had with this question, because they were not fluent with basic material. This example is atypical in that one student had all these problems, but the interview does give a good sample of typical problems that students had in this study.

Class 2, Excerpt of Interview #2, Question 2, K2.

K2. Where x is the real part of z.

P. Both ---.

K2. ....

P. What are you thinking?

K2. I'm trying to again see it as a picture, how we've been doing it, drawing it, the usual way with a line, that's not differentiable, ---. ....[looks through textbook]... I know.......[K2 is apparently referring to the drawings of mappings that were done in the textbook and in class. Presumably, K2 hopes that a picture will help to decide if f(z) is differentiable. This is a carry over strategy from real single variable calculus, that is not very useful even in real calculus on R^2.]

P. What about say, the Cauchy Riemann formula?

K2. Yeh. I know, but the only thing that I don't .., I still have a hard time taking this [$(2z - x)^2$] and writing it in one form that I can do it. Like this is two times, OK, z is --... minus x all squared...

P. OK, z is x plus one, x plus i y.

K2. Yeh.

P. And your trying to find u and v.

K2. Yes. ...

P. OK, so here don't you need 2 (x plus i y)?

K2. Like this?

P. Yes.

K2. Minus x, [all] squared. Squaring functions, I don't have, I don't really understand how to do that. ......

P. I, you square just like you always do, what changes it though is the i, when you multiply the i's together, when you multiply the --, it's negative because of the i's. Since i squared is negative one.

K2. OK, so you just take the, so you say that, x squared plus, this is going to have a minus?

P. Right because of the i squared.

K2. OK. ...So that's differentiable, [K2 is referring to $x^2 + 4ixy - 4y^2$ as being differentiable]

P. So what are u and v for this?

K2. Um, ...--.......that middle term.......if you square this, you get x squared,

P. You have two x and minus, OK, so that's what you have, and square that,

K2. Hmhm.

P. So you have x squared, minus 4 y squared, that's 2 times 2 times the i squared, and then there's a mix term of two i x y plus two i x y, so you get four i x y. [Here we were intervening more than we like to, hoping to get K2 up to the point of applying the C-R conditions]

K2. OK, I get that, but is this considered to be part of u? Oh, this part's the imaginary part?

P. Well, what you have so far is,

K2. Oh, wait a second, x squared minus, 4 y squared would be u, OK. And then this is just v, 4 i x y. ....And now the derivatives, this has to be ..u_x is equal to 2 x, and u_y is equal to minus 8 y. OK, ...2x is equal to four x, [has found partials of v and is applying C-R conditions] and that can't be right....I've done this wrong.

P. What are you thinking?

K2. ...That it's nowhere differentiable. I don't know, that couldn't be the right conclusion though. I don't think I've done this part right. I haven't done this right.

P. No, you're pretty close. Is there any choice of x that would make that,

K2. x equal to zero, and y equal to zero. So,

P. Does y have to be zero?

K2. Oh, x is equal to zero, by those two, 2x is equal to four x, so x equal zero. Oh, this is zero anywhere, ---. For the second part here, it says 4 y is equal to 8 y, so y is equal to zero. And y equal to zero will be satisfied on the real line, so is it just along here?

P. Where are both of these conditions true at the same time?

K2. Outside quadrants one, two three and four, oh, at the same time. x can't be equal to zero, oh, x has to be equal to zero, and y has to be equal to zero, so just at the origin?

P. Yes.

K2. Hm. OK. He was doing this the other day. He did this part here. And without even going to this, he said OK, x equal to zero, all along here, [points to the x axis]

P. This is the y axis.

K2. Oh, sorry, yeh, all along here, isn't differentiable, and then he did this part. This is differentiable. It's all, isn't this whole portion,

P. Um, no, the Cauchy Riemann conditions, both equations have to hold at any point where it's differentiable. So we need both x and y to be zero.

K2. OK.

The solution to this question can be found in appendix 4. In this question we were interested to see if students would conclude that f(z) was not differentiable anywhere, since it is a composition that involves a function that is not differentiable anywhere. This is, of course, not a correct argument. For example, $f(z) = \overline{(\overline{z})}$ is a composition of two nowhere differentiable functions that is entire. We actually expected that most students would just apply the Cauchy-Riemann conditions.

From the interview with K2, we can see a misunderstanding of what to use mappings for (we cannot decide differentiability from inspecting mapping diagrams); difficulties with basic computation (not realizing that complex multiplication preserves the usual distributive law); the belief that if "it's a polynomial in x and y, it's differentiable"; and confusion about how to apply the C-R conditions (both equations have to hold simultaneously). Although the other four students who attempted this question did not have as many problems as K2, two of them did conclude that f(z) was entire, because it was a polynomial. This is a good example of another theme that arose in our study that we have named "Thinking Real, Doing Complex", but which is not discussed in this dissertation (this was a major theme and had to be eliminated for reasons of space limitations).

6.6.2 Class 2, Interview #2, Question 5

Our next question revealed some problems that students had because of insufficient understanding of the polar representation of complex numbers.

Class 2, Interview #2, Question 5 was: where is $f(z) = z^2 + \theta^2$ analytic? To do this question students needed to identify the real and imaginary parts of f(z) in polar coordinates. The solution is given in appendix 4.

Five students attempted this question, using a variety of strategies. Two students converted into Cartesian coordinates, but could not obtain an answer. Two students could not interpret the Cauchy Riemann equations correctly. One said that f(z) was analytic everywhere because it was a polynomial. None of the students recognized that f(z) is multivalued, so that the first step in the solution should be to define a branch of f(z).

6.6.3 Class 2, Interview #2, Question 8

Our next question was designed to see how lack of fluency with symbolic methods would effect later work.

Class 2, Interview #5, Question 8 was: If γ is a circle of radius 2 centered at i, what is $\int_\gamma \frac{|z+i|}{|\bar{z}-i|}dz$?

The solution to this question can be found in appendix 4. Aside from our intended purpose of studying symbolic methods, this question turned out to be interesting because of the absolute value signs.

For starters, none of the four students who attempted this question noticed that the integrand was just one. Two students parameterized γ, to get $\gamma(\theta) = i + 2e^{i\theta}$. One student correctly simplified the integral and got zero. Another calculated $\overline{z} = i - 2e^{i\theta}$, (which is not correct), and got an incorrect answer.

One student began this problem by parameterizing $\gamma(\theta) = i - 2e^{i\theta}$, which is an unusual way to parameterize a circle centered at i, but it is not incorrect, since the minus sign amounts to a phase shift of θ by π. Since we are integrating over the whole circle, the phase shift will not matter.

As we have mentioned, this question was intended to test students' knowledge of symbolic methods, so we were not too interested in the details of computing the integral. We were primarily interested in whether or not students would notice that the integrand was 1 (except at $z = -i$), and therefore that the integral had to be zero.

The last student to attempt this question got the correct answer of zero, analyzing as follows: they found that the integrand was not defined at $z = -i$, which they noticed was on the contour of integration, but then they claimed that the integrand was analytic everywhere else in the complex plane, so by Cauchy's integral theorem, the integral had to be zero. This student was not confident of their reasoning, but when we tried to get them to think about this question correctly, they instead, revealed further misunderstandings, such as, if f(z) is real valued it is analytic.

It is surprising to us that students at this stage in the course would ignore absolute value signs in their estimation of whether or not a function is analytic. Class 2 was specifically given the fact (on several occasions) that a real valued function, defined on a domain D, is analytic if and only if it is constant. So something like $f(z) = |z - b|$ is not going to be analytic anywhere because it is real valued, but not constant.

In any case, we can see that students made several basic errors while doing this question, although some of these errors are somewhat more advanced than representation errors that we are primarily concerned with in this section.

6.6.4 Class 2, Interview #4, Question 9

Class 2, Interview #4, Question 9 was: Find an upper bound for $\left|\int_\gamma z^2 + 2\,dz\right|$, where the integral is to be evaluated on the contour formed by the segments joining the points 1+i, -1+i, -1-i.

The idea here is to find a maximum for the integrand on γ, then multiply the maximum by the length of the path of integration. This product will be an upper bound for the modulus of the integral. There are various ways to find a reasonably efficient upper bound for the integrand. The solution can be found in appendix 4.

Both students who attempted this question got this result. The only difficulty, was that one student spent some time convincing themselves that $|z^2| = |z|^2$. This student did not think this was true at first. Here is another example of how a problem with basic material interfered with later work, and in this case, almost completely obscured the main point of the exercise, which was to study student understanding of the maximum modulus theorem.

6.6.5 Uneven Skill with Basic Material Affects Later Work: Summary

In conclusion, we have noted several types of basic misunderstandings that students had that affected their solutions to advanced questions we asked in this study. Some of these were: θ is multivalued, $|z|$ is real and is not constant, $|z^2| = |z|^2$, polynomials in x and y are not the same as polynomials in z, and the usual distributive law of multiplication of real numbers holds for the complex numbers.

6.7 Conclusion of Chapter 6

In chapter 6 of this thesis we have considered various aspects of the multirepresentations of complex numbers. We discussed the problem of Shifting Representations, the complex number i as a Unit Vector, several aspects of Basic Facts and Calculations, and a final section on Uneven Skill with Basic Material Affects Later Work.

Under Shifting Representations we found that the students we studied were proficient with the Cartesian and polar forms, could translate most expressions from one form to the other fairly accurately, but that nearly half the students studied did not have good judgment about when to translate from one form to the other. We think our results supports the analysis of other researchers in the field of multirepresentations, such as, the Lest, Post, and Behr criteria for understanding representations or the Dreyfus criteria. Finally, we believe most of the students interviewed showed an object understanding of the algebraic and Cartesian vector methods, but no more than process understanding of polar vector methods (using either reification or APOS models).

In the section of this chapter on i as a Unit Vector, we found direct and indirect evidence that students view the complex number i as a unit vector (also 1 is viewed as a unit vector). Most students interviewed firmly believed that the Im z should be represented on the imaginary axis. In addition, we found evidence that students are extrapolating the notion of direction in R^2 onto the complex plane where it is not appropriate.

Under Basic Facts and Calculations we studied topics such as, student recognition that x and y are real in the Cartesian vector form, the cancellation rule, polar forms, and symbolic calculations.

Our last section in this chapter was Uneven Skill with Basic Material Affects Later Work. In this section we discussed several problems that some students struggled with because they were not fluent enough with basic material.

Chapter 7

Summary and Conclusions

7.1 Overview

In this section we give a brief overview of the first six chapters of this thesis. Chapter 6 is summarized more specifically in the next section (section 7.2).

7.1.1 Research Objectives

We studied three complex analysis classes at two British Columbia universities. We found very little Mathematics Education literature on the subject of complex analysis, so our research objectives were relatively broad: 1. Begin recording what occurs in complex analysis classes at British Columbia universities. 2. Survey and catalogue what sort of problems students were having with this subject, paying particular attention to those problems that appeared to be caused by the mode of instruction. 3. Begin the process of identifying those learning difficulties that would be fruitful to further research. In addition to these broad objectives, we were also interested in some specific issues such as, how does insufficient understanding of basic material affect later work, and is there evidence of students making the transition from a process understanding to an object understanding.

7.1.2 Theoretical Framework

We have used a research model consisting of three parts: a teaching model, a learning model and a model of mathematical knowledge. We did not have any control over the teaching model used, since we were an observer in other instructors' classes. The learning model we used has been described by Confrey. Since this model involves radical constructivism, we have reviewed the history of constructivism and radical constructivism as they pertain to mathematics education. We have attempted to use the model of mathematical knowledge suggested by Burton. Finally, in chapter two we have described our expectations before we entered the studied.

7.1.3 History and Epistemology of the Complex Numbers

We examined the epistemology of each of the four representations of complex numbers studied in this research. By the epistemology we mean what the notations mean, what the + signs and × signs mean, et cetera. We also attempted to identify indicators of *process* and *object* understandings of each of the representations. In addition, we examined some properties that the complex plane does not have such as direction and ordering.

We also reviewed the history of complex numbers with the idea of gaining some insights into what kinds of problems students would have. We found that the research community had a very difficult time accepting complex numbers (taking most of three centuries to do so). Even the development of good geometric models of complex numbers did not lead to immediate acceptance of complex numbers. Despite conceptual difficulties all the leading figures in mathematics were fluent at computations and applications with complex numbers. Thus, we are uncertain how the historical record reflects our findings.

7.1.4 Methodology and Data Analysis

Methodology

We studied three complex analysis classes. The first was a pilot project conducted in the Summer semester of 1996 at Simon Fraser University. Our pilot project gave us many good insights into student difficulties as well as aiding our construction of a research method for this project.

Our main study took place in the fall of 1996 again at Simon Fraser University. We did six sets of interviews over the course of the semester, and collected a large amount of data on a wide variety of topics. We chose to focus on the early part of the course for this dissertation.

For our third class, at the University of British Columbia, we conducted two sets of interviews on the material covered in the first part of the course. The data collected from this part of the study enabled us to analyze elementary topics in considerable detail.

Data Analysis

Since complex analysis is relatively uncharted by the mathematics education community, we have used a data driven framework to analyze our data. This framework consists of identifying a central theme that fits with most or all of the data. In this case, we have chosen to consider the data that we have under the theme of Multirepresentations.

The actual conclusions we have drawn using this framework appear in section 7.2.

7.1.5 Data and Analysis - Overview

In chapter 6 we analyzed the data that we collected on our multirepresentations theme. We divided the analysis into four areas: shifting representations, i is a unit vector, basic facts and calculations, and uneven skill with basic material affects later work. The analysis in each of these sections includes one or more of: the problems studied, results from the worksheets, interview excerpts, and analysis of the data. All of the problems used in this study are in appendix 3. The solutions to the problems analyzed in chapter 6 are in appendix 4.

7.2 Conclusions from Our Data and Analysis

We have organized our discussion and conclusions into the categories of chapter 6, so that the reader can readily refer to relevant sections of chapter 6, while reading this discussion.

7.2.1 Shifting Representations

We found that the students we studied had a good grasp of the Cartesian vector forms (for example, $z = x + iy$) and polar forms, but slightly less skill translating from one form to the other. At least half the students studied did not have good judgment about when to shift from one form to another. All shifting was between the algebraic extension and polar vector forms, as we found that students had almost no skill with other forms such as the symbolic form.

Our data supports the criteria identified by others for understanding single representations and translating between two representations. In addition, our data supports inclusion of a further criterion requiring good judgment of when to shift from one representation to another. These criteria can be briefly expressed as follows: 1. Ability to use a single representation. 2. Ability to represent a problem in different representations. 3. Ability to translate between representations. 4. Ability to judge when to shift from one representation to another.

Finally, we have concluded that most students had an object understanding of the algebraic and Cartesian vector forms, but only a process understanding of the polar vector form. Our evidence for this conclusion is largely based on the fact that most students applied prior knowledge when using the simplification algorithm to simplify complex fractions. Since this evidence is admittedly not entirely convincing, further research is needed in this area. In particular, it would be useful to have more information about how an object understanding of each representation manifests itself when working with problems at an elementary level.

7.2.2 i is a Unit Vector

We believe our data supports the conclusion that many student view i as a unit vector (also 1 is viewed as a unit vector). Most of the students interviewed believed that the imaginary part of a complex number should be represented on the imaginary axis. For example, two students specifically mentioned the "imaginary and real unit vectors" in interviews. We also found the most students appear to have a strong sense of direction on the complex plane, for example, most students represented the imaginary part of a complex number on the imaginary axis, even after we questioned them intensively (recall that our data is possibly confounded by the issue of how to interpret Im(z)). These students have apparently extrapolated the notion of direction from R^2. This misconception is not particularly serious, but is very useful to study since the origins of the misconceptions are very clear, so that the problem of correcting this misconception would appear to be clearly defined.

7.2.3 Basic Facts and Calculations

Cartesian Methods

We found that many students had a difficult time consistently applying the fact that in the Cartesian representations, x and y are real, period.

Cancellation

We found that several of the students interviewed were unsure if the usual cancellation rule for simplifying a fraction still applied for the complex numbers. There was considerable confusion about what the structure of the complex numbers is, for example, one student considered both the complex numbers and the real numbers to be scalars.

Polar Methods

Students had several problems applying polar methods: 1. Lack of practice with basic trigonometry, particularly the arctan x function. 2. Little or no understanding of multivaluedness. 3. A tendency to interchange sin θ and cos θ, for example, in Euler's formula or DeMoivre's formula. Generally the modulus did not present problems.

Symbolic Methods

Having described in some detail in chapter five the sort of instruction that students received on symbolic methods, we can make two basic conclusions:

1. The format of presenting a few theorems about symbolic properties, proved by substitution of either the polar or Cartesian vector form for z, appears to be leaving students without any appreciation of the power of symbolic methods or even that they exist.

2. All three of the textbooks studied to some extent take symbolic methods for granted. There were only a few examples on symbolic methods in all three textbooks, and while the textbooks used in the classes we studied have numerous exercises, several of which were assigned, the fact remains that students did not use symbolic methods, even for rudimentary applications, such as, showing that $|\bar{z}|=|z|$.

We have found that almost all the students interviewed did not have access to symbolic methods, with the sole exception that they were able to solve quadratic equations (where they may well be thinking real, rather than using a symbolic approach). We believe that this is a reasonable reflection of the way they have been instructed both in classes and textbooks. Symbolic methods were simply not emphasized in classes or in textbooks. We think this is a mistake since substitution of a vector form is a method of last resort for anybody experienced in complex analysis.

Algebraic Methods

Algebraic methods were certainly not emphasized in the classes we studied, and our limited data indicates that students are learning very little about the algebraic picture of the complex numbers. Of course, a one semester introductory class cannot cover everything, so perhaps questions on algebraic methods should not have been included in our research project.

7.2.4 Uneven Skill with Basic Material Affects Later Work

In this section we surveyed several questions from our data that indicate that students are struggling with more advanced work. For example, on a question asking students where is f(z) differentiable if $f(z) = (2z - x)^2$?, one student attempted to decide by graphing, was unable to find the real and imaginary parts of f(z), did not understand how to calculate the square, concluded that f(z) was entire, since it was a polynomial, and was not sure how to apply the Cauchy-Riemann conditions correctly. In another question we asked students to evaluate the integral $\int_\gamma \frac{|z+i|}{|\bar{z}-i|}dz$, where γ is a circle of radius 2 centered at i. Almost everyone got this correct but no one noticed that the integrand was just one, so that the integral is 0 (recall that the singularity at z = -i is removable). While our conclusions in chapter 6 were quite specific to the questions analyzed, we believe that our data is a small portion of a larger problem. Currently, the course content of all three of the classes we studied only allows for about one week of instruction on basic material. An important question for future research would be to determine if more time spent on instruction on basic topics would pay significant dividends later in the course. For example, if two or three weeks were spent on basic topics, could the time spent be made up later in the course, on the assumption that a thorough grounding in the basics would make subsequent material easier to learn, and thus require less instruction.

In any case, for several of the students studied uneven skill with basic material was a serious impediment to learning later material.

7.3 Directions for Future Research

In chapter 3 we noted that very little work has been done in the area of complex analysis in mathematics education. Thus, our thesis essentially opens a new area of research for the mathematics education community at the university level. Since we are in largely "uncharted" territory there are many directions that research in this field could take, but we believe that several directions for future research are suggested by the data that we have collected and by the analysis in chapter 6.

In the first place, there are many problems of understanding the basic representations that need to be better understood. Some of these are:

> To what extent does not shifting representations indicate lack of understanding as opposed to personal preference?
>
> Can we separate trigonometric difficulties from actual difficulties with the polar representation?
>
> How prevalent is the belief among students that there is direction in the complex plane and how robust is this belief?
>
> How well do students understand basic epistemological questions about the complex numbers such as, what is the meaning of "+" in x + iy?
>
> Our analysis of what constitutes action, process and object understanding within the reification and APOS frameworks of the four representations of complex numbers studied in this thesis is very limited. A great deal of work needs to be done to further clarify how action, process and object understandings manifest themselves.

In addition to research on the representations at which students gained substantial proficiency ($z = x + iy$ and $z = re^{i\theta}$), there are many questions about why students are not acquiring proficiency with symbolic methods and geometric methods. Our data seems to suggest that most student do not realize that symbolic methods exist even though there was significant coverage of these methods in all three classes studied. Was there simply not enough coverage or are there more fundamental obstacles? If so, what are these obstacles and how can we help students overcome them?

We have discussed at some length the data that we collected on how problems that students have with basic material affects more advanced work. This is a big subject and we have only scratched the surface. For example, we did not attempt to correlate the level of understanding within the reification or APOS frameworks with proficiency with more advanced material. We collected some data that suggests that students gain proficiency with basic representations as the course progresses, but there are many more interesting questions that involve studying how understanding changes (or does not change) over the duration of the course. For example, we found that all six students in class 2 had a very robust belief that every polynomial in x and y was analytic. This belief persisted despite the many counterexamples we offered to students throughout the duration of the course. The question is: why?

Aside from the many interesting research questions on basic material, our data indicates that the theme "thinking real, doing complex" is a very significant learning obstacle in complex analysis. This is a major area for further research efforts, since this theme seems to manifest itself in so many ways.

Another major area for mathematics education research in complex analysis at a more advanced level is the whole question of multivaluedness. Multivaluedness is a crucial aspect of complex analysis that is also relatively complicated. Our data suggests that students are absorbing this material with some proficiency, but that there are many questions of understanding to be researched.

Finally, we identified (but did not report in this thesis) many specific themes that warrant further investigation. Some of these are: treating $|z - a|$ as a circle rather than a distance; underestimating how important the c_{-1} term is in a Laurent expansion about a pole; "Is every function analytic?", the difference between the domain of an analytic function and the disk of convergence of the power series for the function expanded about a point in the domain; and the interplay between the path of integration and the integrand of a path integral.

In any case, our conjecture on the feasibility of complex analysis for research in mathematics education at the beginning of this thesis has definitely been shown to be correct. Having surveyed possible research direction, we now turn to the question of what we can say from our data and analysis about the question of how to better teach complex analysis.

7.4 Possible Ways to Help Students Improve their Understanding

We think that our data indicates a number of specific ways that instructors of complex analysis could help students with some of the difficulties we have observed. Within the usual context of a course (as we observed) with a very full schedule of topics, by the phrase "taking more care" (used below), we mean do more examples, assignment questions, and emphasize in class. Thus, we have the following suggestions:

1. Take extra care to insure that students understand that symbolic and geometric techniques are actual techniques for solving problems and not just "interesting facts". For example, we can use the $z = x + iy$ representation to show that $|\bar{z}| = |z|$

for all z. Apparently most students do not realize that they can use this "fact" to solve problems.

2. Instructors could take more care to make sure that students realize that all the field properties of the real numbers (except ordering) extend to the complex numbers, and show how to use this fact.

3. We believe that since complex analysis is largely about what works, many students get the impression that practically any function is analytic. Thus, we suspect that students can be helped substantially by emphasizing, with examples and discovery that many seemingly reasonable functions are not analytic. Perhaps showing students how hard it is for the complex derivative to exist, in effect, demonstrating with the polar form the special way that $f(z)-f(z_0)$ must change as $z-z_0$ changes if *f(z)* is analytic.

4. It is possible that some students might be greatly helped if instructors specifically discussed the techniques from real variable calculus that do not apply in complex analysis. In particular, graphing is not useful, at least in the sense of plotting points. Specifically comparing real variable functions that have one or more derivatives, but are not real analytic, with complex functions might help emphasize how special analytic function are. In class 2 the instructor did mention the well known function $f(x)=\begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$, but this may well be too complicated for most students.

5. The three classes we observed were definitely very focused on the goal of covering at least a few examples of residue theory applied to solving definite real integrals. This has certainly been an important application of complex analysis historically, but with most of these integrals easily approximated numerically, we have to wonder if a review of the goals of the standard first course in complex analysis is not long overdue. We believe that residue theory has become a specialty subject, and that conformal mapping might be a generally more applicable goal of the course.

In this section we have made some suggestions for improved instruction of complex analysis. We have also suggested that the course content be reviewed.

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Appendix 1

Information Sheet

The following information was given to all students in each of the three classes to introduce them to our study, and hopefully encourage them to decide to participate.

FORM #5

SIMON FRASER UNIVERSITY

INFORMATION SHEET

INFORMATION SHEET FOR SUBJECTS

TITLE OF PROJECT: Teaching and Learning Complex Analysis at Simon Fraser University

DESCRIPTION OF PROCEDURES TO BE FOLLOWED:

Collection of data for this study will consist of the following activities:

Observations of students, professor and classroom environment during lectures and tutorials.

Interviews with students and the instructor. **Students will be paid $20 for each interview.**

Free tutoring sessions with students if they wish.

Several brief written questionnaires (included with interviews).

Informal conversations before or after class.

All data will be kept confidential until at least the third week of the 97-1 semester, in effect, until student grades have been clearly established. Names, and any identifying information about students will be kept confidential. In addition, upon completion of this study all data will be destroyed in accordance with university procedures (tapes erased, transcripts shredded, etc.)

QUESTIONS THAT STUDENTS WILL BE ASK: (This list is not exhaustive, but includes the kind of questions that are of interest. Specific mathematical questions may be changed.)

1. What has your mathematics experience been? What factors have contributed to that experience?
2. Can you explain complex division to me (I will provide sample problems)?
3. Ask students to fill-out the Williams [23] questionnaire on limits.
4. What is a complex number?
5. Specific calculation questions, for example, of power series or integrals.
6. Explain what an analytic function is?
7. What is going well for you in this course? What is difficult?
8. Show me how you go about studying a new concept in this course.
9. Questions about singularities, zeros, meromorphic functions, and Laurent series.
10. What do you think mathematics is about?
11. Can you think of any ways that you are able to connect this course to other parts of your life?
12. Applications of Rouchè's Theorem.

QUESTIONS FOR THE INSTRUCTOR:

1. What do you think mathematics is about?
2. What has your mathematics experience been? What factors have contributed to that experience?
3. What is going well for you in this course? What is difficult?
4. What is your philosophy of Mathematical knowledge?
5. What do you think the elements of good teaching are?
6. What do you try to achieve in a class like Math 322?
7. What model of teaching do you use?

THE WILLIAMS QUESTIONNAIRE: (slightly modified)

A. Please mark the following six statements about the limit of a complex function as being true or false:

1. T F — A limit describes how a complex function moves as z moves toward a certain point.
2. T F — A limit is a number or point past which a complex function cannot go.
3. T F — A limit is a number that the w-values of a complex function can be made arbitrarily close to by restricting z-values.
4. T F — A limit is a number or point the complex function gets close to but never reaches.
5. T F — A limit is an approximation that can be made as accurate as you wish.

6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle one) 1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function *f(z)* as $z \to z_0$ is some complex number L.

Appendix 2

Consent Form

The following forms were distributed to students and instructors who chose to participate in our study in accordance with the University's regulations on ethical research.

FORM #2

SIMON FRASER UNIVERSITY

CONSENT FORM

INFORMED CONSENT BY STUDENTS TO PARTICIPATE IN A RESEARCH PROJECT OR EXPERIMENT

The University and those conducting this project subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This form and the information it contains are given to you for your own protection and full understanding of the procedures of the proposed research (specified on form #5). Your signature on this form will signify that you have received a document which describes the procedures of this research project, that you have received an adequate opportunity to consider the information in the document, and that you voluntarily agree to participate in the project in the manner noted by you on this form.

Having been asked by **Peter Danenhower** of the **Dept. of Mathematics and Statistics** of Simon Fraser University to participate in a research project, I have read and understand the procedures specified in the attached document (form #5).

I understand that I may withdraw my participation in this research at any time.

I understand that no information from this study (even of a general nature) will be shared with the professor of the course until the third week of the 96-3 semester, at the earliest. Any information that is shared with the professor, at that time, will be of a general nature, i.e., names or facts that tend to identify subjects in this study will be omitted. In addition, I understand that information gathered during this study will not be used to influence my grade in Math 322-3, or any further courses that I take at Simon Fraser University.

I also understand that I may register any complaint I might have about the research with **Peter Danenhower**, his senior supervisor, **Harvey Gerber**, or with **Len Berggren**, Chair of the Dept. of Mathematics and Statistics of Simon Fraser University.

I may obtain copies of the results of this study, upon completion, by contacting:
Peter Danenhower, Dept. of Mathematics and Statistics, Simon Fraser University, or #210 - 8828 Hudson St., Vancouver, V6P 4N2, ph. 264-9922.

I understand that the research data and analysis of this study may be released for publication, excepting that the names and any information that might tend to identify

(continued on back)

students participating in the study will be omitted. I understand that research data obtained during this study will be destroyed upon completion of the study in accordance with university procedures (tapes erased, transcripts and notes shredded, etc.).

I agree to participate in this study in the ways listed below (please check the appropriate boxes) during the 1996 summer semester at Simon Fraser University:

☐ By being observed during lectures and tutorials.

☐ I agree to be formally interviewed up to six times during the semester. **I will be paid $20 for each interview** if I wish (to be paid at the time of the interview). I understand that these interviews will consist of the kind of questions listed in the information sheet, and may include brief questionnaires. Furthermore, I understand that I can refuse to answer any question, end the interview, and/or any further interviews at any time. I understand that the interviews will be tape recorded, unless I request otherwise.

☐ I agree to participate in the tutoring sessions (office hours) offered several times per week by the researcher (Peter Danenhower) in his office K5911. I understand that I may participate as often as I choose and that tutoring sessions may be tape recorded, unless I ask that they not be recorded. I understand tutoring sessions are being used as a source of information for this research project unless I state otherwise.

☐ I agree that the researcher may use the informal conversations I have with him, for example, while waiting for class to convene, as a source of information for this study.

NAME (please print):____________________________________

ADDRESS:____
PHONE:____________________

SIGNATURE:_____________________
WITNESS:_____________

DATE:__________________

Once signed, a copy of this consent form and a subject feed back form should be provided to you.

FORM #2

SIMON FRASER UNIVERSITY

CONSENT FORM

INFORMED CONSENT BY INSTRUCTORS TO PARTICIPATE IN A RESEARCH PROJECT OR EXPERIMENT

The University and those conducting this project subscribe to the ethical conduct of research and to the protection at all times of the interests, comfort, and safety of subjects. This form and the information it contains are given to you for your own protection and full understanding of the procedures of the proposed research (specified on form #5). Your signature on this form will signify that you have received a document which describes the procedures of this research project, that you have received an adequate opportunity to consider the information in the document, and that you voluntarily agree to participate in the project in the manner noted by you on this form.

Having been asked by **Peter Danenhower** of the **Dept. of Mathematics and Statistics** of Simon Fraser University to participate in a research project experiment, I have read and understand the procedures specified in the attached document (form #5).

I understand that I may withdraw my participation in this research at any time.

I also understand that I may register any complaint I might have about the research with **Peter Danenhower**, his senior supervisor, **Harvey Gerber**, or with **Len Berggren**, Chair of the Dept. of Mathematics and Statistics of Simon Fraser University.

I may obtain copies of the results of this study, upon completion, by contacting:
Peter Danenhower, Dept. of Mathematics and Statistics, Simon Fraser University, or #210 - 8828 Hudson St., Vancouver, V6P 4N2, ph. 264-9922.

I have been informed that the research data and analysis of this study may be released for publication, excepting that my name and any information that might tend to identify me will be omitted. Note, however, that it is possible for a third party to determine the names of the instructors studied in this research, for example, by consulting the 96-2 and 96-3 course schedules.

I understand that research data obtained during this study will be destroyed upon completion of the study in accordance with university procedures (tapes erased, transcripts and notes shredded, etc.).

I agree to participate in this study in the ways listed below (please check the appropriate boxes) during the 1996 summer semester at Simon Fraser University:

☐ By being observed during lectures and tutorials. I also agree to have lectures and tutorials tape recorded.

☐ I agree to be formally interviewed once at the end of the 96-2 semester. I understand that this interview will consist of the kind of questions listed in the "Questions for instructors" section of the information sheet, and that I can refuse to answer any question, and/or end the interview at any time. I understand that the interviews will be tape recorded, unless I request otherwise.

☐ I agree that the researcher may use the informal conversations I have with him, for example, after class, as a source of information for this study.

NAME (please print):______________________________
ADDRESS:____________________________________
PHONE:__________________

SIGNATURE:___________________
WITNESS:_____________

DATE:_________________

Once signed, a copy of this consent form and a subject feed back form should be provided to you.

Appendix 3

Interview Questions

Appendix 3 contains all of the questions that we asked in the interviews for all three classes except for the ethnographic questions. Questions marked with an "*" before the question are questions that we have used in our analysis in chapter 6.

A.3.1.1 Pilot Project (Class 1): First Interview (Early July, 1996)

*Simplify the following, i.e., express them as a + ib, or as $re^{i\theta}$ (whichever you prefer).

*1. $\frac{2+i}{2}$ *2. $\frac{\cdot 2}{1+i}$ *3. $\frac{2+2i}{1+i}$ *4. $\frac{a+ib}{a-ib}$ *5. $\frac{|a+ib|}{a+ib}$

*6. $\frac{-2+2i}{(1+i)^3}$ *7. $-\frac{8}{(\sqrt{3}+i)^6}$ *8. $\frac{2\cdot(1+i)^4}{(-2\cdot\sin 15^0+2\cdot\cos 15^0)^5}$

In questions 9 and 10, by 'describe' I mean give some sort of a picture of what the function looks like. What does it do to points, curves and regions of the complex plane?

9. Describe $f(z)=\frac{z}{1-i}$. 10. Describe $f(z)=\frac{2+i}{z}$

11. If f(z) is a function defined on a region G of the complex plane, what does it mean for f(z) to be holomorphic in G.

12. Suppose f(z) is defined as $f(z)=\frac{1}{i-z}$.

a.) Where is f(z) holomorphic?
b.) Can you find a power series expansion for f(z) about zero? If so, what is it, and what is the radius of convergence?
c.) Does your answer in a) agree with the region of convergence you found in b)? Why or why not?

Alternate Questions (these questions were asked in one interview)

10. Suppose that f: $R^2 \to R^2$ is given by $f(x, y) = (x^2 - y^2, 2xy)$ [So f is (real0 vector valued function defined on the x-y plane].

a.) Describe this function briefly.

b.) Where is f(x, y) continuous?

c.) Is the following integral path independent? $\int_{p_1}^{p_2} (x^2 - y^2)dx + (2xy)dy$

11. Define f(z): $C \to C$ by $f(z) = z^2$.

a.) Where is f(z) continuous? Holomorphic?

b.) Is the following integral path independent? $\int_{z_1}^{z_2} z^2 dz$

c.) Suppose $f(z) = u(x, y) + i\, v(x, y)$, where u and v are real valued, is holomorphic in a region G (in the complex plane). Let g: $R^2 \to R^2$ be defined by $g(x, y) = (u(x, y), v(x, y))$. So g is just the real function with the same components as f(z).

i) Where is g continuous in G? (Thinking of G as a region in R^2)

ii) Is the integral, $\int_\gamma u(x,y)dx + v(x,y)dy$, equal to zero around every closed path in G?

iii) Is the integral, $\int_\gamma f(z)\,dz$, equal to zero for every closed path in G?

A3.1.2 Pilot Project (Class 2): Second Interview (Early August, 1996)

1. Let $A = \{z: R < |z-a| < S\}$ $(0 \le R < S \le \infty)$, and suppose $f \in H(A)$. Then we have for $z \in A$,

$$f(z) = \sum_{n=-\infty}^{\infty} c_n \cdot (z-a)^n$$

where,

$$c_n = \frac{1}{2\pi i}\int_\lambda \frac{f(w)}{(w-a)^{n+1}} dw .$$

Can you explain why this formula for the coefficients is reasonable?

2. Suppose f(z) is defined as: $f(z) = \frac{1}{z(1-z)^2}$.

   a.) Find the Laurent Expansion of f(z) about 1.

   b.) What is the annulus of convergence?

   c.) Why do Laurent Series converge on an annulus?

3. a.) Explain why the zeros of a holomorphic function are isolated.

   b.) Explain why the poles are isolated.

4. a.) Let f(z) = sin z. Is f(z) holomorphic in C?

   b.) Is $\overline{\sin(\bar{z})}$ holomorphic in C?

   c.) Suppose f(z) is holomorphic in C. Is $\overline{f(\bar{z})}$ holomorphic in C? Explain why or why not.

5. a.) Explain why a meromorphic function with no essential singularities in the extended complex plane is a rational function.

   b.) Suppose $f(z) = \frac{(z^2+z+2)(z^3-i)}{(z^4+2)(z^7-6)}$. What is the total order of the zeros in the extended complex plane? What is the total order of the poles in the extended plane?

   c.) Suppose f(z) has a finite number of poles in the extended complex plane and no other singularities. Is the total order of all poles of f(z) equal to the total order of all the zeros? Explain why or why not.

6. Can you use Rouche's Theorem to find out how many zeros $z^9 + 9z - 1$ has in the region $1/2 < |z| < 3/2$?

7. a.) Evaluate $\int_{\gamma(0,4)} \frac{1}{(z^2+1)(z-i)} dz$.

   b.) Evaluate $\int_{\gamma(1,4)} \frac{e^z}{z^4+3z^2+5} dz$.

A3.2.1 Main Study - Fall 1997 (Class 2): Interview Number 1

Part 2

- Simplify the following, i.e., express them as a + ib, or as $re^{i\theta}$ (which ever you prefer).

*1. $\frac{2+i}{2}$;   *2. $\frac{2}{1+i}$;   *3. $\frac{2+2i}{1+i}$;   *4. $\frac{a+ib}{a-ib}$;   *5. $\frac{|a+ib|}{a+ib}$;

*6. $\frac{-2+2i}{(1+i)^3}$;   *7. $-\frac{8}{(\sqrt{3}+i)^6}$;   *8. $\frac{2(1+i)^4}{(-2\sin(\frac{\pi}{12})+2i\cos(\frac{\pi}{12}))^5}$

In questions 9 and 10 I mean give some sort of a picture of what the function looks like.

9. Describe $f(z)=\frac{z}{1-i}$.   10. Describe $f(z)=\frac{2+i}{z}$.

*11. Show that if $|z|=1$, then $|1-z\overline{w}|=|z-w|$.

Part 3

A. Please mark the following six statements about complex numbers as being true or false:

1. T F   A complex number is a vector in the plane.
2. T F   A complex number is a number of the form a + ib, where a and b are real numbers, and $i^2 = -1$.
3. T F   A complex number is a zero of a polynomial with real coefficients.
4. T F   A complex number is a number of the form (x, y), where x and y are real numbers.
5. T F   A complex number is an element of the complex number field.
6. T F   A complex number is a number of the form re^i.

B. Which of the above statements best describes a complex number as you understand it? (Circle one)   1 2 3 4 5 6 None

C. Please describe what you understand a complex number to be.

A3.2.2 Main Study - Fall 1997 (Class 2): Interview Number 2

Part 2

A. Please mark the following six statements about the limit of a complex function as being true or false:

1. T F A limit describes how a complex function moves as z moves toward a certain point.
2. T F A limit is a number or point past which a complex function cannot go.
3. T F A limit is a number that the w-values of a complex function can be made arbitrarily close to by restricting z-values.
4. T F A limit is a number or point the complex function gets close to but never reaches.
5. T F A limit is an approximation that can be made as accurate as you wish.
6. T F A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

B. Which of the above statements best describes a limit as you understand it? (Circle one) 1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a limit to be. That is, describe what it means to say that the limit of a function f(z) as z z_0 is some complex number L.

Part 3

Answer the following questions. Remember whatever solution you find or however you think about these problems is perfect! Feel free to use the textbook.

*1. Find all solutions to $|z - i| = |z + 1|$.

*2. Suppose $f(z) = (2z - x)^2$, where x is the real part of z. Where is f(z) differentiable?

3. Where is $f(z) = 3\bar{z}^2 + z + 4$ continuous?

4. Find all solutions of $|e^{i\theta} - 1| = 1$

*5. If $\theta = \arg z$, where is $f(z) = z^2 + \theta^2$ analytic?

6. If $z = re^{i\theta}$, where is $f(z) = re^{2i\theta}$ differentiable?

7. If $z = re^{i\theta}$, where is $f(z) = r\theta e^{-i\theta}$ continuous?

8. Find all solutions of $2|z - 3| = |z - 6|$.

9. Suppose $f(z) = z^2$, if $Re(z) \geq 0$, and $f(z) = 0$, if $Re(z) < 0$. Where is f(z) differentiable?

10. Where is $f(z) = x^2 - y^2 - 2ixy$ analytic?

11. Let $f(z) = 0$ if $|z| < 1$, and $f(z) = z^2 - 1$ if $|z| \geq 1$. Where is f(z) analytic?

12. Suppose f(z) is analytic on D and real valued. Explain why or prove that f(z) is constant.

A3.2.3 Main Study - Fall 1997 (Class 2): Interview Number 3

Part 2

A. Please mark the following six statements about analytic functions (of a complex variable) as being true or false.

1. T F	Any polynomial in x and y will be entire.
2. T F	An analytic function is a function that is differentiable everywhere in an open disk.
3. T F	Analytic functions are special because the limit formed by the derivative has to hold from any direction.
4. T F	Any rational function of e^z is entire.
5. T F	Complex multiplication (division) is key to understanding analytic functions.
6. T F	An analytic function satisfies the Cauchy - Riemann conditions in the domain of the function.

B. Which of the above statements about analytic functions is most accurate?

1 2 3 4 5 6 None

C. Which of the above statements about analytic functions is most useful to you?

1 2 3 4 5 6 None

D. Please describe, in a few sentences, what you understand an analytic function to be.

Part 3

Do each of the following questions anyway you wish.

1. Simplify $\frac{3+6i}{1+2i}$.

2. Find the harmonic conjugate of $u(x, y) = e^y \cos x$.

3. If $z = re^{i\theta}$, where is $f(z) = \theta e^{-ir}$ analytic?

4. Verify that $|e^{iz}| = |\cos z + i \sin z|$

*5. Find the real part of $\dfrac{(i\cos\theta + \sin\theta)^5}{(\cos 2\theta + i \sin 2\theta)^2}$.

6. Find all solutions to $|3z + 1 - i| = |3z - 4|$

7. If $f(z) = (x - y)^3 + i(x + y)^3$, where is f(z) differentiable?

8. Find all solutions to $|z - i| = |\bar{z} - i|$.

9. If f(z) is defined as: $f(z) = 0$ for $\mathrm{Im}(z) < 0$, and $f(z) = z^2$ for $\mathrm{Im}(z) \geq 0$, where is f(z) analytic?

*10. If $|z| = 2$, show that $|2 - \overline{zw}| = |zw - 2|$, for all w.

11. Find all solutions to sin z = 3. What is cos z for these values of z?

A3.2.4 Main Study - Fall 1997 (Class 2): Interview Number 4

Part 2

A. Please mark the following six statements about branches of multivalued functions as being true or false.

1. T F   A branch of a multivalued function restricts the values to make a single valued function.
2. T F   A branch cut of a multivalued function is a ray or curve which the function can't cross.
3. T F   A branch cut of a multivalued function is a restriction of the domain to make it single valued.
4. T F   A branch of a multivalued function is a restriction on the argument of z to a fixed interval less than or equal to 2 in length.
5. T F   The branches of a multivalued function have the same value (in the extended complex plane) at a branch point.
6. T F   Branch points and points on a branch cut are points at which the multivalued function is not analytic.

B. Which of the above statements best describes the properties of branches of multivalued functions as you understand them? (Circle at most one from 1-4, and at most one from 5 and 6)

1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a branch of a multivalued function to be. Include branch cuts and branch points in your description.

Part 3

Do the following problems any way you wish.

1. Show by direct calculation that $\left|\int_0^\beta f(\gamma(t))\gamma'(t)dt\right| \le \int_0^\beta |f(\gamma(t))\gamma'(t)|dt$,where $f(z) = z + 2$, and $\gamma(t) = (1 + i)t$.

2. Find $\int_\gamma \sin z\,dz$, where γ is the segment from 0 to 1 on the x axis, followed by the vertical segment from 1 to 1 + i.

3. For which closed contours, γ, does $\int_\gamma z + \frac{1}{z}dz = \int_\gamma \frac{1}{z}dz$?

4. Find $\int_\gamma e^z\,dz$, where γ is the square with vertices at 0, 1, 1 + i, and i.

5. Find $Log(-1 + i)^3$. Is this equal to $3Log(-1 + i)$?

6. If a, b, and c are complex numbers is $a^c \times b^c = (ab)^c$?

7. Find all solutions of $\cos z = 3i$.

8. Find $\int_\gamma x\,dx + y\,dy + i(x\,dy - y\,dx)$, where γ is the upper semicircle connecting 0 and 4 + 4i.

*9. Find an upper bound for $\left|\int_\gamma z^2 + 2dz\right|$ where the integral is to be evaluated on the contour formed by the two segments joining the points 1 + i, -1 + i, and -1 - i .

A3.2.5 Main Study - Fall 1997 (Class 2): Interview Number 5 and 5A

Part 2

A. Please mark the following six statements about contour integrals of complex functions as being true or false.

1. T F — A contour integral of a complex function, f(z), is roughly equal to f(z) times the length of the path.
2. T F — If the integral of the complex function f(z) over every closed contour in a domain D is zero, then f(z) is analytic.
3. T F — If f(z) is a complex function, analytic on and inside a contour, , then for all z inside , f(z) is determined by it values on .
4. T F — The modulus of a contour integral is always less than or equal to the maximum of the integrand on the contour times the length of the contour.
5. T F — Contour integrals are useful for calculating the value of a function, for example, using Cauchy's integral formula.
6. T F — Contour integrals are defined in terms of two real valued line integrals.

B. Which of the above statements best describes the way you think of a contour integral? Circle one.

1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand a contour integral to be.

Part 3

Do the following questions any way you wish.

1. Find $\int_\gamma 3x^2 + iy\,dz$, where γ is the square formed by the points 0, 1, 1 + i, and i.

2. Find $\int_\gamma \frac{1-4iy+x^2y}{z}dz$, where γ is the unit circle centered at the origin.

*3. Show that for $\gamma(t) = e^{it}, 0 \le t < 2\pi$, $\int_\gamma \frac{1}{z} - \bar{z}dz$ is zero. Is $\frac{1}{z} - \bar{z}$ analytic?

4. Suppose f(z) is entire, and has the property that for some real constants, a and b, $\neq 0$, $f(z + a) = f(z)$, and $f(z + ib) = f(z)$. Use Liouville's Theorem to show that f(z) is constant. (A function with this property is called doubly periodic.)

5. Let D be the whole complex plane. What's wrong with the following argument: Since $f(z) = 1/z^2$ has an anti-derivative, the integral of f(z) over any closed contour is zero, so by Morera's theorem, f(z) is analytic in D, i.e., $1/z^2$ is entire?

6. Suppose γ is a closed contour in the complex plane that contains complex numbers a

and b in its interior. Can you use the <ML technique to find an upper bound for the following integral? What do you conclude? $\int_\gamma \frac{1}{(z-a)^2(z-b)}dz$.

7. Is the results of problem 6 changed if we changed the integral to $\int_\gamma \frac{1}{(z-a)^2|z-b|}dz$? Why or why not?

*8. If γ is a circle of radius 2 centered at i, what is $\int_\gamma \frac{|z+i|}{|\bar{z}-i|}dz$?

9. Find the following integral for $\gamma(t) = 1 + i + 2e^{it}$, $0 \le t \le 2\pi$, $\int_\gamma \frac{e^{2z}}{z(z-5i)(z-\pi)}dz$.

Interview 5A differed from interview 5 in the following questions:

5. Let $f(z) = \frac{1}{i-z}$. i) What is the domain of f(z)? ii) Expand f(z) into a power series about z = 0. What is the radius of convergence? Is it the same as the Domain of f(z)? Why or why not?

7. Suppose z_0 is real, and let $f(z) = \sum c_n(z-z_0)^n$ be defined for $|z - z_0| < R$. Is $f(\bar{z})$ defined? What is it? Is it analytic? What about $\overline{f(\bar{z})}$?

A3.2.6 Main Study - Fall 1997 (Class 2): Interview Number 6

Part 2

A. Please mark the following six statements about Taylor series as being true or false.

1. T F	If f(z) is defined by a Taylor series, convergent on a disk D, then f(z) is analytic on D.
2. T F	If f(z) is analytic in a domain D, and z_0 is in D, the power series for f(z), expanded about z_0, is convergent for all z in D.
3. T F	If the power series for f(z) diverges at z_0, then f(z) is not analytic at z_0.
4. T F	The product of the Taylor series for f(z) and g(z), expanded about z_0, converges at each point where f(z) and g(z) are analytic.
5. T F	If f(z) and g(z) have Taylor series expanded about the point z_0, with radii of convergence R_1 and R_2, respectively, then the Taylor series for f(z)g(z), expanded about z_0, has radius of convergence R_1.

6. T F If f(z) is entire, then the Taylor series for f(z), expanded about any point, converges in the whole complex plane.

B. Which of the above statements best describes the behaviour of analytic functions and their Taylor series. Circle one.

1 2 3 4 5 6 None

C. Please describe in a few sentences what you understand about convergence of Taylor series of analytic functions.

Part 3

Answer the following questions anyway you wish.

1. Let $f(z) = \frac{1}{i - z}$. i) What is the domain, D, of f(z)? ii) Expand f(z) into a power series about z = 0. What is the radius of convergence? Is the domain of convergence of the series the same as D? Why or why not?

2. Suppose $f(z) = \sum c_n (z - z_0)^n$ is a Taylor series and z_0 is real. If z is inside the disk of convergence is $f(\bar{z})$ defined? Is $f(\bar{z})$ analytic at z? What about $\overline{f(\bar{z})}$?

3. Suppose f(z) is entire, and m is a postitive real number such that the $\lim_{z \to z_0} \frac{f(z)}{(z - z_0)^m}$ exists and is not zero. Can m be 1/2? what do you conclude?

4. Let $f(z) = \frac{1}{\cosh z}$. Find the Taylor series of f(z) about 0. What is the radius of convergence?

5. Why do power series converge on disks, and Laurent series converge on annuli?

6. Suppose g(z) is an entire function, such that g(x) = sin x. Use power series to show that g(z) = sin z.

7. Find the Laurent series for $f(z) = \frac{\cosh z}{(z + i)(1 + z)}$, expanded about -1.

A3.3.1 Main Study - Spring 1998 (Class 3): Interview Number 1

*1. What is the real part of the following?

a. $4 + 2i$

b. $3 - (1 + i)i$

*2. If we use the $z = (x, y)$ representation for complex numbers, which of the following are correct statements? (Circle the correct ones)

a. $3 + 6i = (3, 6)$

b. $3 + 6i = 3i(-i, -2)$

c. $3 + 6i = 3i(-i, -2i)$

d. $3 + 6i = (3, 6i)$

e. $3 + 6i = (2 + i, -i + 5)$

f. $3 + 6i = (1+2i)(1, 2)$

g. $3 + 6i = 3(1, 2)$

h. $3 + 6i = (2 + i, 1 + 5i)$

*3. a) Put the following into a + bi form.

i) $\dfrac{2+4i}{1+2i}$

ii) $\dfrac{-6+3i}{1+2i}$

b.) Three students did the following problem three different ways. Which of them are correct? (Circle the correct ones)

i) $\dfrac{6-9i}{2-3i} = \dfrac{6-9i}{2-3i} \cdot \dfrac{2+3i}{2+3i} = \dfrac{12+27}{4+9} = \dfrac{39}{13} = 3$

ii) $\dfrac{6-9i}{2-3i} = \dfrac{6-9i}{2-3i} \cdot \dfrac{2+3i}{2+3i} = \dfrac{12-11i+9i+27}{4+9} = \dfrac{39-2i}{13}$

iii) $\dfrac{6-9i}{2-3i} = \dfrac{3(2-3i)}{2-3i} = 3$

c. Put the following into a + bi form.

i) $\dfrac{-4+2i}{2-i}$

ii) $\dfrac{6+2i}{-3-i}$

4. a) Simplify the following expressions, given that $z = (x, y)$.

i) $(3, 2)(1, 4)$

ii) $\dfrac{(3,2)}{(1,4)}$

*b) Simplify the following given that $z = re^{i\theta}$.

i) $3e^{i\pi} \times 5e^{i\frac{\pi}{2}}$ ii) $3e^{i\pi} \div 5e^{i\frac{\pi}{2}}$ iii) $3e^{i\pi} + 5e^{i\frac{\pi}{2}}$

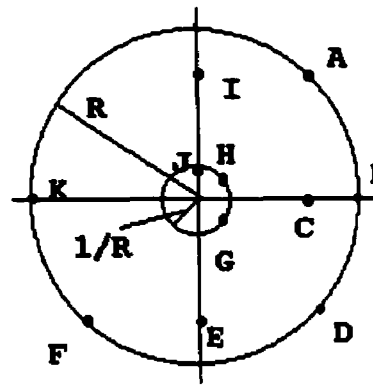
5. a) Mark the following six statements about complex numbers as true or false.

1) A complex number is a point in the complex plane.
2) A complex number consists of a real part, a, and an imaginary part, b, that are added together (with a factor of i in front of b) to get $z = a + ib$.
3) A complex number is the endpoint of an arrow that starts at the origin of the complex plane.
4) A complex number is an ordered pair, such as, $z = (x, y)$.
5) A complex number is a vector in the complex plane.
6) If Rez is x, and Imz is y, the $z = x + iy$.

b) Which of the above six statements best describes how you think of a complex number? (Circle one) 1 2 3 4 5 6

c) Describe how you think about complex number in your own words.

*6. Use the diagram to fill in the blanks below. Select a point for each item. The large circle has radius R, and the small circle has radius 1/R. The first one is done for you (and tells you where z is).



i) z is__A__

ii) |z| is _____

iii) z bar is _____

iv) 1/z is _____

v) Im (z bar) is_____

vi) Re z is______

vii) |z bar| is ______

ix) 1/(z bar) is______

x) Im z is ______

xi) Re (z bar) is ______

xii) -z is ______

*7. a) Find the solutions of the following equations:

i) $z^2 + 2iz - 1 = 0$ ii) $z^2 + (1 + i)z + i = 0$

b) Are your solutions complex conjugates? When are the solutions of a quadratic equation complex conjugates? More generally, given that w is a root of the P(z), what conditions on P(z) assure that w bar is also a root?

8. Find the solutions of the following equations:

a) $z^5 = i$ b) $z^4 = -2\sqrt{2}(1 + i)$

A3.2.1 Main Study – Spring 1998 (Class 3): Interview Number 2

1. In this question a, b, c, and d are real numbers.
 a) For which b is it true that, i) $|ib| = |2i|$ ii) $|3 + ib| = |3 + 2i|$

b) Find all b and c for which i) $|3 + ib| = |3 + ic|$ ii) $|b - 2i| = |c - 2i|$

c) Find all combinations of a, b, and c for which i) $|ib| = |a + ic|$ ii) $|3 - ib| = |a + ic|$.

d) Does $|a + ib| = |c + id|$ always imply that $|a| = |c|$, and $|b| = |d|$? Why or why not?

*2. a) Is $3i < 5i$?

b) Suppose we define a lexicographic ("dictionary") order on the complex numbers as follows: $a + ib \le c + id$ if and only if either $a < c$, or $a = c$, and $b \le d$. Does this ordering preserve the usual rules of multiplication and addition on both sides of an inequality?

c) Find the real and imaginary parts of $\dfrac{(2+i)-3i+(1+i)i}{(2-i)i-2+3i}$.

3. Which of the following functions are continuous? Why or why not?

a) $f(z) = z^2 + 2$ b) $f(z) = (\bar{z})^2 + 3$

c) $f(z) = 0$ if $\text{Im } z \ge 0$, and z^2 if $\text{Im } z < 0$.

d) $f(z) = 0$ if $\text{Re } z \ge 0$, and z^2 if $\text{Re } z < 0$.

e) $f(z) = z^2 - 1$ if $|z| \ge 1$, and 0 if $|z| < 1$.

f) $f(z) = e^{-\frac{1}{z^2}}$ if $z \ne 0$, and 0 if $z = 0$.

*4. a) For which z is $|\bar{z}| = |z|$? Explain.

b) For which z is $|\bar{z} + 1| = |z + 1|$? Explain.

c) For which z is $|\bar{z} - i| = |z + i|$?

d) For which z is $|iz + 1| = |\bar{z} + i|$?

e) Where is $f(z) = \dfrac{|\bar{z}-1|}{|2z-2|}$ analytic?

5. a) Does $[\sin(4\theta) + i\cos(4\theta)] = (\sin\theta + i\cos\theta)^4$?

   b) Does $[\sin(n\theta) + i\cos(n\theta)] = (\sin\theta + i\cos\theta)^n$ for all n?

6. a) Find all roots of $z^4 = -2i$

   b) Find all roots of $z^4 + z^2 = -2i$

Appendix 4

Interview Questions and Solutions

The solutions given in this appendix are simply solutions, and are not necessarily the solutions we expected students to give. Only the solutions to questions used in our analysis in chapter 6 are included in this appendix.

A.4.1.1 Pilot Project (Class 1): First Interview (Early July, 1996)

Simplify the following, i.e., express them as a + ib, or as $re^{i\theta}$ (whichever you prefer).

1. $\frac{2+i}{2}$ 2. $\frac{2}{1+i}$ 3. $\frac{2+2i}{1+i}$ 4. $\frac{a+ib}{a-ib}$ 5. $\frac{|a+ib|}{a+ib}$

6. $\frac{-2+2i}{(1+i)^3}$ 7. $-\frac{8}{(\sqrt{3}+i)^6}$ 8. $\frac{2\cdot(1+i)^4}{(-2\cdot\sin 15^0+2i\cdot\cos 15^0)^5}$

Solutions:

1. $\frac{2+i}{2}=\frac{2}{2}+\frac{1}{2}i=1+\frac{1}{2}i$

2. $\frac{2}{1+i}=\frac{2(1-i)}{(1+i)(1-i)}=\frac{2(1-i)}{2}=1-i$

3. $\frac{2+2i}{1+i}=\frac{2(1+i)}{(1+i)}=2$

4. $\frac{a+ib}{a-ib}=\frac{z}{\bar{z}}=\frac{z^2}{|z|^2}=\frac{a^2-b^2}{a^2+b^2}-\frac{2ab}{a^2+b^2}i$

5. $\frac{|a+ib|}{a+ib}=\frac{|z|}{z}=\frac{\bar{z}}{|z|}=\frac{a-ib}{\sqrt{a^2+b^2}}$

6. $\frac{-2+2i}{(1+i)^3}=\frac{2\sqrt{2}e^{\frac{3\pi i}{4}}}{\left(\sqrt{2}e^{\frac{\pi i}{4}}\right)^3}=1$

7. $-\frac{8}{(\sqrt{3}+i)^6}=\frac{8e^{i\pi}}{\left(2e^{\frac{i\pi}{6}}\right)^6}=\frac{8}{64}=\frac{1}{8}$

8. $\frac{2\cdot(1+i)^4}{(-2\cdot\sin 15^0+2\cdot i\cos 15^0)^5}=\frac{2\left(\sqrt{2}e^{\frac{i\pi}{4}}\right)^4}{2^5 i^5 e^{\frac{i5\pi}{12}}}=\frac{e^{\frac{i\pi}{12}}}{4}$

A.4.2.1 Main Study – Fall 1997 (Class 2): Interview Number 1

Part 2

Simplify the following, i.e., express them as a + ib, or as $re^{i\theta}$ (which ever you prefer).

1. $\frac{2+i}{2}$; 2. $\frac{2}{1+i}$; 3. $\frac{2+2i}{1+i}$; 4. $\frac{a+ib}{a-ib}$; 5. $\frac{|a+ib|}{a+ib}$;

6. $\frac{-2+2i}{(1+i)^3}$; 7. $-\frac{8}{(\sqrt{3}+i)^6}$; 8. $\frac{2(1+i)^4}{(-2\sin(\frac{\pi}{12})+2i\cos(\frac{\pi}{12}))^5}$

A.4.2.2 Main Study – Fall 1997 (Class 2): Interview Number 2

Part 3

Answer the following questions. Remember whatever solution you find or however you think about these problems is perfect! Feel free to use the textbook.

1. Find all solutions to $|z - i| = |z + 1|$.

Solution:

The solution to this equation is the set of all points that are equidistant between i and -1.

So z = t(1-i), for all real t.

2. Suppose $f(z) = (2z - x)^2$, where x is the real part of z. Where is f(z) differentiable?

Solution:

In this case, $f(z) = (x + 2iy)^2 = x^2 - 4y^2 + 4ixy$. Taking partial derivatives, and applying the Cauchy-Riemann equations, $u_x = 2x$, $u_y = -8y$, $v_x = 4y$, and $v_y = 4x$. So we require, $2x = 4x$, and $8y = 4y$. Thus, f(z) is differentiable only at (0, 0) (the partials are continuous everywhere).

5. If $\theta = \arg z$, where is $f(z) = z^2 + \theta^2$ analytic?

Solution:

The first thing we have to do with this function is define a branch, because the θ^2 term makes this function multivalued. So we choose $0 \le \theta < 2\pi$. Then it is convenient to use the polar form of the C-R conditions: $ru_r = v_\theta$, and $rv_r = -u_\theta$. Then $f(z) = r^2\cos 2\theta + \theta^2 + ir^2\sin 2\theta$, so $u = r^2\cos 2\theta + \theta^2$ and $v = r^2\sin 2\theta$. Computing the partials, $ru_r = 2r^2\cos 2\theta$, $u_\theta = -2r^2\sin 2\theta + 2\theta$, $rv_r = 2r^2\sin 2\theta$, and $v_\theta = 2r^2\cos 2\theta$. So we require $2r^2\cos 2\theta = 2r^2\cos 2\theta$, and $2r^2\sin 2\theta = 2r^2\sin 2\theta + 2\theta$. The first equation is an identity, but the second only holds if $\theta = 0$. Thus, $f(z)$ is differentiable on the ray, Re $z > 0$, but is nowhere analytic.

A.4.2.3 Main Study – Fall 1997 (Class 2): Interview Number 3

Part 3

Do each of the following questions anyway you wish.

5. Find the real part of $\dfrac{(i\cos\theta + \sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^2}$.

Solution:

$\dfrac{(i\cos\theta + \sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^2} = \dfrac{ie^{-5\theta}}{e^{i4\theta}} = ie^{-9i\theta} = i(\cos 9\theta - i\sin 9\theta) = i\cos 9\theta + \sin 9\theta$. So the real part is $\sin 9\theta$.

10. If $|z| = 2$, show that $|2 - \overline{zw}| = |zw - 2|$, for all w.

Solution:

This question has a mistake in it: $|2-\overline{zw}| = |\overline{2-zw}| = |2-zw| = |zw-2|$ for all w and z. We do not need the restriction on z given in the question.

A.4.2.4 Main Study – Fall 1997 (Class 2): Interview Number 4

Part 3

Do the following problems any way you wish.

9. Find an upper bound for $\left|\int_\gamma z^2 + 2dz\right|$ where the integral is to be evaluated on the contour formed by the two segments joining the points 1 + i, -1 + i, and -1 - i .

Solution:

The idea in this problem is to find a reasonable upper bound for $z^2 + 2$, then multiply by the length of the path. Since the entire path is contained in the disk centered at the origin of radius $\sqrt{2}$, and the maximum of an analytic function occurs on the boundary, we can take the maximum value of $z^2 + 2$ to be the maximum on the circle centered at the origin of radius $\sqrt{2}$, namely 2 + 2 = 4. The length of the path is 2 + 2 = 4, so an upper bound for the modulus of the integral is 4 squared, or 16.

A.4.2.5 Main Study – Fall 1997 (Class 2): Interview Number 5

Part 3

Do the following questions any way you wish.

3. Show that for $\gamma(t) = e^{it}$, $0 \le t < 2\pi$, $\int_\gamma \frac{1}{z} - \bar{z}dz$ is zero. Is $\frac{1}{z} - \bar{z}$ analytic?

Solution:

For the contour given the integrand is identically zero, since $\frac{1}{z} = \bar{z}$ on the unit circle. So the integral is zero. We cannot conclude that the integrand is analytic (Morera's theorem), because we have only checked one very special contour. In fact, the integrand is nowhere analytic (using the Cauchy-Riemann conditions).

8. If γ is a circle of radius 2 centered at i, what is $\int_\gamma \frac{|z+i|}{|\bar{z}-i|} dz$?

Solution:

The integrand is just one, except at the point z = -i where there is a removable singularity (the numerator and denominator are equal, since they are the moduli of complex conjugates). Since the singularity is removable, we can ignore it, and Cauchy's integral theorem still applies. So the integral is zero, since f(z) = 1 is entire.

A.4.3.1 Main Study – Spring1998 (Class 2): Interview Number 1

1. What is the real part of the following?

   a. 4 + 2i   b. 3 - (1 + i)i

Solution:

a) The Re(4 + 2i) = 4. b) 3 - (1+i)i = 3 - i + 1 = 4 - i, so Re (3 - (1+i)i) = 4.

2. If we use the z = (x, y) representation for complex numbers, which of the following are correct statements? (Circle the correct ones)

   a. 3 + 6i = (3, 6)   e. 3 + 6i = (2 + i, -i + 5)

   b. 3 + 6i = 3i(-i, -2)   f. 3 + 6i = (1+2i)(1, 2)

   c. 3 + 6i = 3i(-i, -2i)   g. 3 + 6i = 3(1, 2)

d. $3 + 6i = (3, 6i)$ h. $3 + 6i = (2 + i, 1 + 5i)$

Solution:

In the $z = (x, y)$ representation, there is no i, so any expression that contains i is automatically incorrect. Thus, only part a and part g are correct.

3. a) Put the following into a + bi form.

i) $\frac{2+4i}{1+2i}$ ii) $\frac{-6+3i}{1+2i}$

b.) Three students did the following problem three different ways. Which of them are correct? (Circle the correct ones)

i) $\frac{6-9i}{2-3i} = \frac{6-9i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{12+27}{4+9} = \frac{39}{13} = 3$

ii) $\frac{6-9i}{2-3i} = \frac{6-9i}{2-3i} \cdot \frac{2+3i}{2+3i} = \frac{12-11i+9i+27}{4+9} = \frac{39-2i}{13}$

iii) $\frac{6-9i}{2-3i} = \frac{3(2-3i)}{2-3i} = 3$

c. Put the following into a + bi form.

i) $\frac{-4+2i}{2-i}$ ii) $\frac{6+2i}{-3-i}$

Solution:

Part a), i) $\frac{2+4i}{1+2i} = \frac{2(1+2i)}{1+2i} = 2$. ii) $\frac{-6+3i}{1+2i} = \frac{3i(1+2i)}{(1+2i)} = 3i$.

Part b), i) is correct, ii) is incorrect (the cross terms in the numerator have been calculated by adding: -9i + 2 "=" -11i, and 6 + 3i "=" 9i), and iii) is correct.

Part c), i)

4. b) Simplify the following given that $z = re^{i\theta}$.

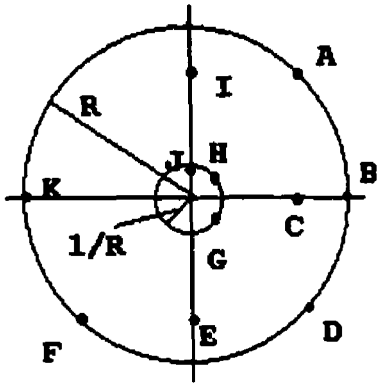
i) $3e^{i\pi} \times 5e^{i\frac{\pi}{2}}$ ii) $3e^{i\pi} \div 5e^{i\frac{\pi}{2}}$ iii) $3e^{i\pi} + 5e^{i\frac{\pi}{2}}$

Solution:

b) i) $3d^{i\pi} \cdot 5e^{i\frac{\pi}{2}} = 15e^{i\frac{3\pi}{2}}$ ii) $3e^{i\pi} \div 5e^{i\frac{\pi}{2}} = 0.6e^{i\frac{\pi}{2}}$

iii) $3e^{i\pi} + 5e^{i\frac{\pi}{2}} = -3+5i = \sqrt{34}e^{i(\pi + \text{Arc}\tan\frac{5}{-3})}$

6. Use the diagram to fill in the blanks below. Select a point for each item. The large circle has radius R, and the small circle has radius 1/R. The first one is done for you (and tells you where z is).



i) z is__A___

ii) |z| is _____

iii) z bar is _____

iv) 1/z is _____

v) Im (z bar) is____

vi) Re z is______

vii) |z bar| is ______

ix) 1/(z bar) is______

x) Im z is ______

xi) Re (z bar) is ______

xii) -z is ______

Solution:

ii) $|z| = R$; iii) $\bar{z}$ is D; iv) 1/z is G; v) Im($\bar{z}$) is -C (no point for this); vi) Re(z) is C;

vii) $|\bar{z}|$ is R; ix) 1/($\bar{z}$) is H; x) Im(z) is C; Re($\bar{z}$) is C; -z is F.

7. a) Find the solutions of the following equations:

i) $z^2 + 2iz - 1 = 0$ ii) $z^2 + (1 + i)z + i = 0$

b) Are your solutions complex conjugates? When are the solutions of a quadratic equation complex conjugates? More generally, given that w is a root of the P(z), what conditions on P(z) assure that w bar is also a root?

Solution:

Part a) i) $z^2 + 2iz - 1 = (z + i)^2 = 0$, $z = -i$. ii) $z^2 + (1 + i)z + i = (z + 1)(z + i) = 0$, so $z = -1$, or $z = -i$.

Part b) The roots are not complex conjugates in either case. The solutions of a quadratic equation will be complex conjugates (we regard real roots as their own complex conjugates) if and only if P(z) = cQ(z) where c is any constant, and Q(z) is a quadratic with real coefficients. The general case will be similar, since any polynomial with real coefficients can be factored into polynomials with real coefficients, and of degree at most equal to 2, in effect, quadratics. This theorem is not too hard to prove, but we will only sketch it here: If the roots are conjugates then the complete factoriztion of the polynomial (in C) will have factors, such as, $(z-a)(z-\bar{a})$ which can be multiplied out to a quadratic with real coefficients. So we will get a product of polynomials with real coefficients (up to the constant c). On the other hand if Q(z) has real coefficients, then it can be factored into factors that are of at most degree 2, so the roots will either be real or complex conjugates.

A.4.3.2 Main Study – Spring 1998 (Class 2): Interview Number 2

2. a) Is $3i < 5i$?

b) Suppose we define a lexicographic ("dictionary") order on the complex numbers as follows: $a + ib \le c + id$ if and only if either $a < c$, or $a = c$, and $b \le d$. Does this ordering preserve the usual rules of multiplication and addition on both sides of an inequality?

c) Find the real and imaginary parts of $\dfrac{(2+i)-3i+(1+i)i}{(2-i)i-2+3i}$.

Solution:

Part a) $3i < 5i$ makes no sense, in the usual sense of an ordering, because we cannot define the relationship between 0 and i: If $i < 0$, then $i^2 > 0$, but $-1 < 0$. Or if $i > 0$, then $i^2 > 0$, but $-1 < 0$. Thus, we cannot define an order that preserves the usual rules when multiplication is applied.

Part b) We can order the complex numbers as a set with the lexicographic ordering, but this ordering does not follow consistent rules for multiplication: for example, by this ordering $0i \le 1i$, but then (multiplying by i) $0 \le -1$, so, apparently, we need to change the order when multiplying by i. But then, since $1 \le 2$, we would have $2i \le i$, which contradicts the ordering rule. We conclude that the lexicographic ordering does not work (if we want to use multiplication).

4. a) For which z is $|\bar{z}| = |z|$? Explain.

b) For which z is $|\bar{z} + 1| = |z + 1|$? Explain.

c) For which z is $|\bar{z} - i| = |z + i|$?

d) For which z is $|iz + 1| = |\bar{z} + i|$?

e) Where is $f(z) = \dfrac{|\bar{z}-1|}{|2z-2|}$ analytic?

Appendix 5

Two Complete Interview Transcripts with Worksheets

In this appendix we have included two complete interview transcripts, including photocopies of the sections of the worksheet that go with each question. We have selected these interviews simply as average in length, and one from class 2 and one from class 3. Here a series of periods "....." means there was a pause of length proportional to the number of periods: 2 or 3 periods means a few seconds, half a line of periods means a few minutes. Also a series of "***" or "---" means untranscribed phrase. The number of stars or dashes is equal to the approximated number of words missing.

Interview #3, Class 2, Fall 1996

Part 1

Question 1

1) How do you like university (as a learning and social environment)?

F. How do I like university? As a learning and social, well it depends, like, um, I don't like SFU as a social environment very much, it could be better, ****, much better, but that's OK, um, as a learning environment, I don't know, it ** what I've done I guess, **. I suppose I like it, else I wouldn't be here. [laughs] I'm not doing this because I have to, because I want to, so suppose I like it in that respect. Socially, I don't think this particular university **.

Question 2

2) Do you agree that the university mathematics environment puts women at a disadvantage? Why or why not?

F. Do I agree that the university mathematics environment puts women at a disadvantage? Oh, well, no I don't think so, really. Um, in some ways I guess, that I agree. In my earlier courses, when I decided to go on, I just got looks of surprise, from many, you know, um, but besides that, you know, I found that there's so many more women involved in it now than, I think, ever before, so I don't think it does at all.

Question 3

3) Do you agree that the university mathematics environment supports male attitudes? Why or why not?

F. Do I think it supports male attitudes? What kind of male attitudes? [laughs] What do you mean by that?

P. OK, it's hard to describe, I may not find the best way, but, um, it's the idea that, um, the mathematics environment has been set by social patterns that they act out. They aren't necessarily destructive patterns or bad patterns, it's just the way men are.

F. Unhun. Oh, I don't know. *** [laughs] Well, I suppose it would in the fact that it's suppose to be, you know, mathematics is a relatively traditional kind of thing. You know, in that respect I would say so. But at the same time, the more females get into it, and the more there's, um, females instructing it and that kind of thing, then there's going to be, you know, not nearly so much of that, I guess. But, um, I don't know, it does, but it doesn't. I think it does, but ** word **, I suppose, a little more, and as well just because of the way math is, ***, kind of like any science ***.

P. Can you think of any specifics that come to mind?

F. Like how it supports male attitudes?

P. Yes.

F. Oh, my goodness. I can't ***. This is one of those **, you know. I'd say that, uh, ...um, I don't know, it sounds to me like, um, if you're talking about male attitudes, then, I suppose, you're, I don't really know. Really, that's a tough question. Like we, right now we found that, uh, in class it's kind of nice to find that the text books seems to be written more with males in mind somehow. But how it was, because I was just taking a text, that we, you know, I don't find the text very easy to follow a lot of times. But males seem to be able to look at them and read them, and it's no problem. You know, so I don't know if that has something to do with it, that kind of thing, because they're mostly going to be written by males, so it's being aimed at people at xx, if they're saying that you know that males think differently than females, or whatever, then it could be in that respect. And that could ****.

P. OK, so

F. Be even more specific?

P. Yeh, can you tell me specifically about the text, how it's different?

F. Well, I find I learn best by examples,

P. Learn mathematics,

F. I learn best by examples, and I find that the textbook, the earlier level textbooks are find, there's lots of examples, but as you move-up the textbooks get a lot more explanation based as opposed to example based. And so I have a hard time taking the explanations and turning it into, and doing a question, from that. I'd rather have an example, because then, I learn a lot better that way.

P. OK. all right you've picked out something and can you explain, I'm not challenging you at all, I'm just asking.

F. Oh, no, that's all right.

P. So can you explain how you verify that these are male patterns or male attitudes?

F. Well, just from the discussion with other people, even I, but, um I just find that most of the females I've been talking to learn better by example, and the male attitude, I don't know if it may be that, um, I don't know, like I don't even know how, it's just that when it's explain to me, it just seems to, like to xx know explain, it just seems to go on and on and on, and, ah, and the same thing could easily be done, you know by showing an example, and explaining an example, as opposed to explaining a lot more abstractly, I think. I know if that's a male thing, you know that males are, you know learn easier that way, by taking something that's a little more abstract to, you know, to doing examples, and perhaps in that sense, *****, but that's just from talking with other people, they have the same kind of knack. Is that specific enough?

P. Yes.

F. OK! [laughs]

Part 2

A. Please mark the following six statements about analytic functions (of a complex variable) as being true or false.

1. T F Any polynomial in x and y will be entire.
2. T F An analytic function is a function that is differentiable everywhere in an open disk.
3. T F Analytic functions are special because the limit formed by the derivative has to hold from any direction.
4. T F Any rational function of e^z is entire.
5. T F Complex multiplication (division) is key to understanding analytic functions.
6. T F An analytic function satisfies the Cauchy - Riemann conditions in the domain of the function.

B. Which of the above statements about analytic functions is most accurate?
1 2 3 4 5 6 None

C. Which of the above statements about analytic functions is most useful to you?
1 2 3 4 5 6 None

D. Please describe, in a few sentences, what you understand an analytic function to be

Section A

Question 1

F. Oh no, ** a number of things. OK. Um, OK, will yeh, a polynomial is entire because it's analytic everywhere, right, so, if I remember correctly.

Question 2

F. A function that is differentiable everywhere in an open disk. Um, ... oh, that's right, oh, right, sure.

Question 3

F. Are special because.... Yeh, that's true.

Question 4

F. Um, a stupid question, what's a function of e to the z?

P. It's not stupid at all.

F. [laughs] OK. So will you tell me, or [laughs]

P. Yes. Rational functions are, they're functions of the form q of z over p of z, where q and z [q and p] are polynomials.

F. OK.

P. And so a rational function of e to the z, you put e to the z in instead of z.

F. Oh.

P. That's a polynomial in **

F. In z, oh, so, oh. Oh, OK. Will if it's still a polynomial it would be entire, although it's e to the z. It's a polynomial though?

P. It's a polynomial in e to the z over a polynomial in e to the z. So it's something like, here's an example e to the two z, that's e to the z all squared, plus 2i times e to the z, plus 4 all over e to the z all to the sixth power minus 1.

F. Oh, OK. So would that be entire? Oh, ...well the bottom could wind up being zero or something, couldn't it? So then it wouldn't be, because it wouldn't exist at that point. That's what I'd say, so.

Question 5

F. Is key to understanding analytic functions....I'd say false because, um, I'm not quite sure why, it just doesn't make good sense.

Question 6

F. An analytic function satisfies the Cauchy - Riemann conditions, yeh, well it's analytic at that point, anyway. So I bet you they're all true, right?

P. No, some of them are false.

F. Oh, wow! Because last time they were all true. [laughs]

Section B

F. Which of the above statements about analytic functions is most accurate? ....Ah, I have to choose just one?

P. Yeh.

F. OK, ...I'd say two, and

Section C

F. Then I'd say the most useful to me would be the Cauchy-Riemann conditions.

Section D

F. What do you understand an analytic function to be? Do I say it out loud or write it down.

P. Either one.

F. Well, OK, I just have to think about it for a second. This is all stuff from the midterm. Based on what I can remember is that an analytic function is, wait let me get this, um, ...I do actually know this, I just can't think of it right now. An analytic function is at a point and it's differentiable at any points around that particular point in the plane, and um, and that the Cauchy Reimann conditions hold at *** [the point] where it's differentiable, around that point. Does that make sense? And if it's analytic at every point, on the function, or whatever, then it's entire. That's what I remember. And I should have said something about continuity too, I just can't remember where it goes. Well that's what the Cauchy Riemann stuff I'm talking about there ***. But that xx analytic, so. Yeh, I guess, *** understand the practical end of things. All right, OK, that was a good question, **.

Part 3

Do each of the following questions anyway you wish.

---

1. Simplify $\dfrac{3 + 6i}{1 + 2i} \dfrac{(1-2i)}{(1-2i)}$

$$\frac{3 - 6i + 6i + 12}{1 + 4} = \frac{15}{5} = 3$$

Question 1

F. Oh, god, OK, simplify, what do you mean simplify, like just,

P. Put it,

F. Into another form?

P. Put it into a form simpler than that.

F. Then that.

P. a plus ib form or r e to the i theta.

F. Oh, OK. Well, I guess I'll try the good old complex conjugate of this......How about that.

P. OK.

F. Right? Yes?

P. Yes.

---

2. **Find the harmonic conjugate of $u(x, y) = e^y \cos x$**

$v(x,y)\ e^y \cos x$

$U_x = -e^y \sin x = V_y \qquad U_x = V_y$

$U_y = -V_x$

$\int -e^y \sin x \, dy$

$v(x,y) = -e^y \sin x + \phi(x) \qquad U_y = e^y \cos x = -V_x$

$V_x = -e^y \cos x + \phi'(x) \qquad V_x = -e^y \cos x$

$v(x,y) = -e^y \sin x + C$

Question 2

F. Oh, I can't remember how to do this. .....Oh, geez...OK, um, u x equals ... v y, is that right? It's ** cosine x, negative sine x, I always get those mixed-up, OK. Um, and so u x equals, um v y, u y equals negative v x, that's right isn't it?

P. Yes.

F. OK, so then that means this equals u y, so I get, I used that and equate this....then that's just a constant, that's just *** itself **. OK, should I go on? ..... OK, Oh, **. .....And then I have to take u y, because I want to get that, I am suppose to compare them. Is that right?

P. Yes.

F. Yeh, OK. It's all coming back now. OK so u y is going to equal um, e to the y cos x, and then that would ***. So we get cos, minus e to the y cos x. So, hm, ...Then using this, OK, so I've got this v x equals, oh, I can't remember,...I think *** say this is the same as that. I have to find out what this is. I know what I have to do, I just can't remember how to do it. xx, um,

P. What's v x?

F. V x is negative e to the y cosine x.

P. And what'd you get for v x here?

F. Oh, so it's, oh, thank you. [laughs] OK. negative e to the y cosine x plus ..., and so these are the same, so that means that's [phi prime] just a constant, right because that goes to, right, yeh.

P. That's constant?

F. So, now, just wait. Oh, no, ah, oh, that's zero, so that means that's a constant?

P. Yes.

F. Yeh, OK. So then that means that um, is that right? Do I ** the whole thing, no, OK.

P. Just for the tape, you were saying phi prime is zero,

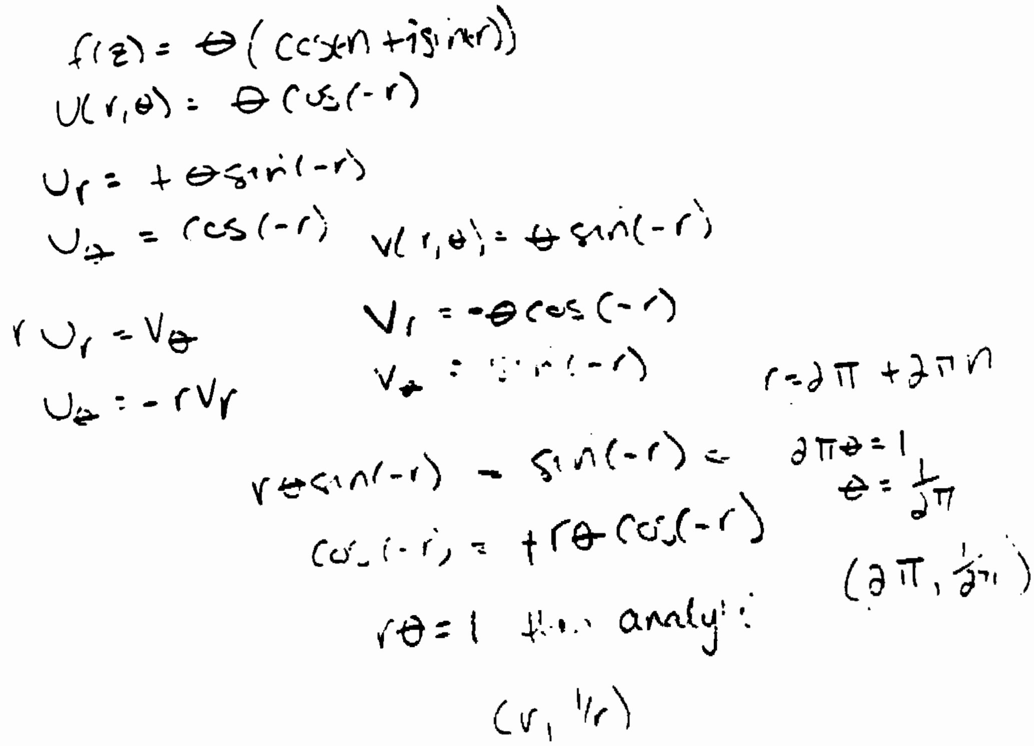
F. So therefor phi is a constant.

P. OK.

F. OK.

---

3. If $z = re^{i\theta}$, where is $f(z) = \theta e^{-ir}$ analytic?



Question 3

F. Um, if z equals r e i theta, where is ***? .....Hm, ... So I just want to find where this is analytic basically.

P. Yes.

F. So you're just saying this is, this is just saying that it's in polar form **, oh.

P. Yes.

F. OK, so that means that, um this f of z equals theta times cosine r ..., so since that's negative, I'm just going to say this is negative r. OK, um, so u of r, theta equals .....I couldn't remember when there's a negative sign in there if that makes cosine negative, so I'll just leave it [laughs]. Just in case you're wondering why I'm doing that, because I know one of them is going to wind up being negative and one of them positive. ***. Um, OK so, u r is ****. And v r is ....OK, it's um, oh, gee, correct me if I'm wrong here, I can't remember this part, u r, is it u r equals one over, I thought it was something really unusual.

P. ***.

F. OK, I ** it was like one over r, or something, xx, OK. r u r equals v theta,

....OK, um r times u r is r theta sin equals ..., and cos .. equals r ... OK, so, ah, well this is true when r and theta both equal one, ***, so it's analytic when r equals one and theta equals ...these have to be equal for this to work. So the only way that's going to work is if r and theta both equal one, or r times theta equals one.

P. r times theta.

F. OK. So r theta if that equals one then it's analytic, right?

P. Are there any other possibilities?

F. I bet there are since you are asking. Ah, oh, if cosine negative r is equal to zero. So I mean, r is a value that it's going to equal zero, then both these would be equal to zero, right? Does that make sense?

P. Say that again.

F. Sorry, if it ends up both side are equal to zero, so sin of some value of r is going to equal zero, and so ***. Um, sin of zero is zero, isn't it, and cos of one is zero, is that right?

P. Remember r can't be zero,

F. Oh, no, because that this point, oh. Oh, so it has to ..., so zero, so it would be any multiple of, would it be two pi?

P. OK.

F. So if r equals two pi here, then it would be like any multiple of that, right?

P. OK.

F. So this one would be, well, cosine, cosine one is zero, isn't it? If we let r be one, r be negative one?

P. No,

F. That doesn't make sense.

P. r is two pi plus a multiple, so what's cos of that?

F. What's cosine of two pi?

P. Yes.

F. Um, oh, it's one. OK. so then if I, I'm ***. So, if um, so, I want to get zero so, is that,

P. Is one there and one there.

F. So you're saying I have to put two pi in, right? What I'm I going to equal if I put 2 pi **, oh so the whole thing doesn't work. So i have to find an r that satisfies all of these, but there isn't one. Is that right? Like I made this one work, for that, but then that one doesn't work.

P. Are you sure?

F. Oh, OK, um, [laughs] If I put two pi in here these aren't equal.

P. What do you get if you put two pi in there?

F. This is one.

P. OK, cos of two pi is one.

F. This is one, and then would be two pi theta.

P. OK, equals?

F. One.

P. OK.

F. Oh, OK, if two pi theta equals one, then um, theta would equal one over two pi, oh so it would be two pi and one over two pi, right?

P. OK, what about this case and this case? Are they different?

F. Oh, look at that they are the same thing. OK, so, [laughs] what do you know? So then it would be basically like say r and one over r. Like that?

P. Yes.

F. Or I could say like one over theta, theta. Isn't that the same thing? So, basically that would be the points, where it's analytic.

P. So go back to the beginning, what was the question?

F. Where is it analytic?

P. And what is your answer?

F. It's analytic when theta equals one over r.

P. Is that right? That's not exactly what youv'e shown.

F. What have you shown? ...What do you mean what have I shown? Like what did I do?

P. Yes, when you do the Cauchy Riemann conditions, what does that show.

F. What does it show? Oh, what does that show? It shows that it's differentiable at those points. And because of that it's analytic, right? No?

P. Is there any difference between differentiability and analyticity?

F. I'm sure there is since you're asking. [laughs] I've just got to think about it. Is there any difference between those two? Um, I don't know, my guess would be not....because if it's differentiable then, if it's differentiable then it's going to be analytic, right, isn't that right? Because it's not differentiable unless the Cauchy Riemann conditions hold.

P. OK, you've definitely found where it's differentiable.

F. I don't know.

P. You have.

F. Oh, I have, OK, OK. So that's where it's differentiable, these points right?

P. Yes.

F. OK, so then I want to find where it's analytic, right? OK, is there something really obvious I'm missing here?

P. No.

F. OK, just checking. Um,.....I don't know, it's analytic where ever it's differentiable, that's what I would say.

P. OK, let's go on, I'll tell you at the end.

---

4. Verify that $|e^{iz}| = |\cos z + i \sin z|$

$$\left|\frac{e^{iz}+e^{-iz}}{2} + i\,\frac{e^{iz}-e^{-iz}}{2i}\right|$$

$$= \left|\frac{e^{iz}+e^{-iz}+e^{iz}-e^{-iz}}{2}\right| = \left|\frac{2e^{iz}}{2}\right| = |e^{iz}|$$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{-y}e^{ix}| = e^{-y}$$

$$\sqrt{\cos^2 z + \sin^2 z} = 1$$

$$e^{ix} = (\cos x + i\sin x)e^{-y}$$

$$= e^{-y}\cos x + i e^{-y}\sin x$$

$$\sqrt{(e^{-y}\cos x)^2 + (e^{-y}\sin x)^2}$$

$$\sqrt{e^{-2y}\cos^2 x + e^{-2y}\sin^2 x}$$

$$\sqrt{e^{-2y}(\cos^2 x + \sin^2 x)}$$

$$\sqrt{e^{-2y}}\,\sqrt{\cos^2 x + \sin^2 x}$$

$$e^{-y}$$

Question 4

F. Oh, gee, OK, um, ...verify it? Well I just kind of know that.......

P. What are you thinking?

F. What am I thinking? I'm thinking I just know this. [laughs] I can't think of what I could possibly do to verify it. Short of, I don't even know. If I .... I assume that this is true.

P. Yes.

F. Yeh. So that now.......Can I do it the other way around?

P. Yep.

F. Yeh. OK, so maybe I'll try it by putting cos z and sin z in the other form, with the e to the i z. Would that be a good idea? And then I add it all together and then I would hope to get that.

P. OK.

F. OK. Ah, so cos z is, oh gee, I can't remember, e to the.....I can't remember what it is. ***, that's minus, is it just e to the minus z? I can't remember, is that right? I can't remember what it is.

P. Right, you don't want both ***.

F. So that one's positive?

P. Yes.

F. It doesn't matter which one? And so in this case,

P. One of those is positive.

F. And that is a **. OK, then these cancel, the i's cancel on this side, and we're left over two, so, ...I lost the i, where's the i? ***, what did I do wrong? I missed something, is that i? Is it e to the i z?

P. e to the i z.

F. OK. There's one of them where there's no i, *** there's no i **. OK, so that makes a little more sense. OK, so two e to the i z equals e to the i z. That's what I'd do. I'm doing it backwards.

P. OK.

F. I suppose, actually there's something else I could do with it.

P. So, let me give you an alternative, and you can critique this.

F. Wow.

P. Instead of doing that, you say e to the i z is equal to e to the i times x plus iy,

F. Is equal to, OK.

P. And so that's e to the minus y times e to the i x, right, because it's i times x,

F. Oh, right, OK.

P. And the modulus of that is just e to the minus y, this is one [modulus of e to the i x].

F. I never thought that I should just take the modulus of that. OK.

P. And then the other side is, we just square, so it's equal to the square root of cos square z plus sin squared z,

F. OK.

P. And that's one.

F. Right.

P. So, what did I do wrong? This is e to the minus y, and that's one.

F. Yeh, they're different. Oh, ...don't you have to include the imaginary part in the ***, or the, I see not, that part you go from here to here. .....so,...

P. So which one of these do you think is correct?

F. Oh, that one looks fine. [That the modulus is one]

P. That one's good?

F. Yes. It's going from this to that, [the modulus of e to the minus y times e to the i x equals e to the minus y] because if you went, so you're taking the modulus of that, you have to, you just took that part, you just took the real part.

P. No, remember that's in polar form, it's r e to the i theta.

F. Oh, OK,

P. This is theta [x is theta], and r is e to the minus y.

F. Oh, OK, ...OK, well, ...But you're taking the modulus of the whole thing, not just of that.

P. OK, well how would you find the modulus?

F. Well, that's what I'm trying to figure out. And I would try to write it something like that form **, and do it that way.

P. OK, what's e to the i x?

F. e to the i x is cosine, oh, OK. e to the i x is cosine x plus i sin x, and that's time e to the negative y. And so then e to the minus y cos x, and so that's the real part and that's the imaginary part. And so then you square that, ...and hopefully that comes out to one. [laughs]

P. ***?

F. Well, I squared the whole thing.

P. Oh, OK.

F. You want me to finish what I was doing. So that's going to be ....this is one, wait a minute, I'm getting e to the negative y. So there must be something wrong with this one.

P. You're right. e to the minus y is definitely correct.

F. OK, so there's something wrong with that. Oh, because z has a real and imaginary part, so I have to figure out what those are, right. OK, do I actually have to do it, or is it enough to just show it.

P. That's ***.

F. OK, I'm kind of annoyed with myself for xx, [laughs] OK,

P. That step is ***.

F. Yeh, I ***, but, yeh, that makes sense though, that once you put x plus i y in that form, um and jumbo it around it's going to be different **. OK, so that's my **.

5. Find the real part of $\dfrac{(i\cos\theta + \sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^2}$

$$\frac{(-\cos\theta + i\sin\theta)^5}{i(\cos 2\theta + i\sin 2\theta)^2}$$

$$\frac{+i(\cos\theta - i\sin\theta)^5}{(\cos 2\theta + i\sin 2\theta)^2} = +i\frac{(e^{-i\theta})^5}{(e^{2i\theta})^2}$$

$$= \frac{i(e^{-5i\theta})}{(e^{4i\theta})}$$

$$i\, e^{-5i\theta - 4i\theta}$$

$$i e^{-9i\theta}$$

$$i(\cos(9\theta) - i\sin(9\theta))$$

$$i(\cos(9\theta) + \sin(9\theta))$$

$$\sin(9\theta)$$

Question 5

F. Find the real part of that. ....*** midterm.

P. So what's your first thought about that question?

F. It's a lot of multiplying. [laughs] That's my first thought. Or, um, you know because this is just, oh it's not. .... This looks like it'd be something along the lines of that [Euler's formula], but it's not the same thing. The i's not right. But if I, say I multiply this by i, right, in the brackets here, um the whole thing, **, then i would get, negative cosine theta plus i sin theta. But then I still have that negative there, so ***. Is that on the right track, to do something along those lines?

P. Sounds good to me.

F. But then I just, well...........Well if I just multiply this by i, right, I mean it has to be to the fifth power. That's why I'm just having a few problems getting my mind around **. Yeh, you know, I can't just multiply by i, because there's going to be..., you know, you have to multiply the bottom by i also. Is there some i missing here or, ...Am I just generally confused. Otherwise I'll just multiply the whole stupid thing out, but I'm sure that's not the line of thinking you want. [laughs]

P. Fortunately, ***.

F. I'm sure it has something to do with getting this in a form such that I can write e to the whatever. That's right isn't it?

P. Yes.

F. Yeh, but I can't just figure out how I would do that, um with out, um, ...Can I just do that, if I multiply by i, it's going to end up going all in the brackets **, is that what you want me to do. I can do that? You're just going to let me do it. OK, well, we'll try that and see what happens. OK, so it's going to be negative cosine theta plus i sin theta......

P. What exactly did you do?

F. I multiplied top and bottom by i.

P. Top and bottom by i.

F. I can do that can't I. I can put that into the bracket, right? I can do that? That's what I'm not sure of either.

P. What happens?

F. Because it ends up being to the fifth power.

P. OK, and what's i to the fifth?

F. Oh, That's an odd one, it's just i, oh, so that's the same difference, so it doesn't matter then.

P. OK.

F. Right, OK, ...so then, this can't just be e to the i theta, because there's a negative in front of cos theta, so ***. Or, ...if, isn't cos theta minus i sin theta e to the negative i theta?

P. That's true.

F. OK, say I take negative one out of that [laugh] I'm doing all sorts of illegal stuff. All these simple rules that I just forget.......Oh, but one over i is negative i isn't it? xx, OK, so then I can just ***. .....Ah, e to the negative i theta to the fifth over,

P. Sorry what was that step there?

F. Oh, which one?

P. You canceled the i.

F. No, I took the i up, so one over i equals negative one, right? And I have a negative there,

P. OK.

F. And so, negative one times negative one there is positive, that's what I did.

P. OK, but what happened to the i?

F. It's gone, because it equals, oh, it equals negative i, oh, xx, thank you. Sorry, OK, ah, actually that's going to be e to the .....i two theta, OK, um, I'm still trying to find the real part, aren't I? Oh, right, I forgot what I was trying to do.....I'm not sure that's very helpful, myself, what's it do, we still have i over here.

P. What is different now?

F. Well, I'm trying, I forgot that I'm trying to find the real part [laughs] I just remembered, and ah, I've now got it in a form where, um, I'm just going to put it back in to sines and cosines, um, unless, could this equal, e to the negative 5 i theta, I can just take that in there can I?

P. Yes.

F. OK. And this will be to the four i theta, OK, if this is being divided, I have to subtract, is that right? So I have i times e to the, this is just to the negative nine, OK. And then i times cos nine theta minus i sin nine theta, that would be i cosine, oops, doesn't matter, ah, yes. Well, I found the real part but I don't know if I'm correct. So the real part was sin nine theta I got.

P. Yes.

F. Is that right? Wow!

P. Well done.

F. Way to go! Just got a little confusing. All right. [tape change]

---

6. Find all solutions to |3z + 1-i| = |3z - 4|

$$|3(x+iy)+1-i| = |3(x+iy)-4|$$
$$|3x+3iy+1-i| = |3x+3iy-4|$$
$$|3x+1+i(3y-1)| = |3x-4+3iy|$$
$$\sqrt{(3x+1)^2+(3y-1)^2} = \sqrt{(3x-4)^2+(3y)^2}$$
$$\sqrt{9x^2+6x+1+9y^2-6y+1} = \sqrt{9x^2-24x+16+9y^2}$$
$$9x^2+9y^2+6x-6y+2 = 9x^2+9y^2-24x+16$$
$$30x-14 = 6y$$
$$y = 5x - 14/6$$
$$(x, 5x - 14/6)$$

Question 6

F. ..... plus one, ...I'm trying to find what z has to be equal, or what x and y have to be equal?

P. Either.

F. OK, I'm going to put in x plus i y.........***......Well, x can be, ...well, x can be anything I suppose, it doesn't matter what ** goes in here.

P. Right.

F. But the y that has to be, so then, oh, why don't l if I take the modulus of both sides, isn't absolute value, sort of like that's what the length so I always forget that I can actually do that.

P. Yes.

F. OK, um, ...I'm not sure that helped me any, um, ...Now I don't know what, OK, ****....,

P. Sorry, what's the problem?

F. I don't know! [laughs] I don't, I am not quite sure what to do about this, like I want to make them equal to each other, so I, um,

P. You have the square root of that whole thing, and you have the square root on the other side,

F. Oh, I could square both sides. Um, OK, ...these all cancel, um, OK, and, xx, ...OK, I ***. Ah, well short of saying, you know, x, and then this thing as the y, right, that would be it. Wouldn't that be all the solutions? No? Yes?

P. Yes.

F. So that just an odd looking answer.

P. What does this look like? What kind of solution is this?

F. What kind of solution set? Like what does it look like, or?

P. Geometrically?

F. Ah, ...So if I were, I guess, ***, um, would it be, I'm not very good at xx things in **. Um, well it's just going to be, um, ...I don't know, I'd have to like draw, like I'd have to sketch points and draw, and see what it looked like. Do you want me to do that?

P. No.

F. I just thought those other things xx, just knowing. So can we go on?

P. Yes.

---

8. Find all solutions to $|z - i| = |\bar{z} - i|$

$$|x + iy - i| = |x - iy - i|$$

$$|x + i(y-1)| = |x + i(-y \pm 1)|$$

$$\sqrt{y^2 + (y-1)^2} = \sqrt{x^2 + (-y-1)^2}$$

$$x^2 + (y-1)^2 = x^2 + (-y-1)^2$$

$$y^2 - 2y + 1 = y^2 + 2y + 1$$

$$-2y = 2y$$

$$-y = y$$

$$y = 0$$

Question 8

P. Why don't you do this one.

F. OK.

P. Try 8 now.

F. Is that an i?

P. This is, that's i, z minus i equals z bar minus i.

F. ....Um, **, ...** these cancel, and you get negative two y equals two y. This is odd, negative y equals y, so that means that, oh, that makes sense, um, so the only time negative y equals is y is zero. Is that right? So y is zero..... So y equals zero, x can just be anything, can't it? ***.

P. ***.

F. Well, y equals zero here [inside absolute value signs, after substituting z = x plus iy] means they're going to be equal so, x can be anything. Right?

P. Do you agree?

F. That's what I'm saying.

P. OK.

F. [laughs] Is that correct?

P. Yes.

F. Yes. OK.

P. So, when you got that, you said something made sense, what was that?

F. Oh, because there was a bar on z, and that means one of the y's is going to be negative on one side and positive on the other side.

P. Because of the z bar?

F. Yes. It's the complex conjugation. Right?

P. OK.

F. OK, now I have to do this thing, it doesn't look good, does it?

P. Well, let me just ask you, when you look at this, do you have any geometric picture at all?

F. Ah, probably not. Um, well if I wrote it out into uh, no that doesn't work either.....no. I told you I'm awful at that, I really don't think geometrically.

P. OK.

F. No I have no idea what that look's like. I guess it would ah,... I suppose it would just be on the x axis. Because of that [y = 0], right?

P. Yes.

F. ***?

P. Yes.

F. OK. I'm not sure if that helps. [laughs]

P. OK.

F. OK, time to try this one.

---

9. If f(z) is defined as: f(z) = 0 for Im(z) < 0, and f(z) = z^2 for Im(z) ≥ 0, where is f(z) analytic?

$(x+iy)(x+iy)$

$x^2 + 2ixy - y^2$

$u(x,y) = (x^2 - y^2)$ $\quad v(x,y) = 2xy.$

$u_x = 2x$ $\quad v_x = 2y$

$u_y = -2y$ $\quad v_y = 2x$

$u_x = v_y$ ✓ $\quad 2x = 2x$

$u_y = -v_x$ ✓ $\quad -2y = -2y$

Question 9.

[F. OK, now I have to do this thing, it doesn't look good, does it?, comment made above as part of interview during question 8]

F. ....You know, I'm not even sure how to approach it.

P. OK, to start with do you understand what this is? It's a function that's defined differently on,

F. On the two parts. Yes.

P. OK. So what's this part saying?

F. It's saying when the imaginary part of z is less than zero then f of z is going to be equal to this.

P. OK, so in the lower half plane it's always going to equal zero.

F. Yes, so basically, I want to do it separately for each and see where the whole thing is analytic?

P. Yes.

F. Is that the idea? OK, so um, well this part then, well, that's analytic everywhere, because, yeh, the real and imaginary part are, they're both, well the imaginary part can't be equal xx? Uh, ...Well, if f of z is just equal to zero, then it's analytic. OK, so then that's analytic everywhere in that domain. And this one is........[finds u and v for f(z) = z squared, and checks C-R conditions] OK, 2x equals 2x, that's fine, so in this case, then it's striking me that, y would have to equal zero, then. [F. first got 2y = - 2y, for second C-R condition] or else they can't be, because see again I got that y equals negative y. So then, what have I done?

P. U is?

F. U is x squared plus y squared.

P. U is x squared minus y squared.

F. OK, u y is, then it's true. So that's true. So this is analytic everywhere also, on this, so it's analytic on both regions. Right?

P. On both?

F. It's analytic on all of z plane.

P. OK.

F. You're going to tell me if I'm right or not. [laughs] Well, that's what I'd say, because if it's analytic all on this part for imaginary z, and I just found that, because these hold at all points x and y, that this is analytic at all points, then it's going to be analytic on the entire imaginary **.

P. OK, are you sure?

F. Sure as I'm going to be, yeh. [laughs]

P. OK, so what happens on the x axis?

F. On the x axis?

P. What if z was eight?

F. Was eight? Then that's what x is. So x would be eight.

P. Sorry, if z is eight?

F. Oh, if z is eight then x is, no the real part is eight, and there's no imaginary part.

P. OK.

F. And so the ah, so that would be where, z is eight. So that would use, xx on the x axis?

P. Well, I'm just picking z equal to eight.

F. OK, if z we're equal to eight, then that would be, um, I know what xx look like, ***.

P. What would the limit be of f of z, as z approaches 8, from the upper half of the plane?

F. Well the limit as z approached 8 would be, good question, I don't know.

P. So what's f of z do in the upper half plane?

F. In the upper half plane. It would be 16.

P. 16?

F. No, it would be 64. [laughs] OK, yeh, it would be 64.

P. OK, and what's the limit as z approaches 8 from the lower half?

F. That 's going to be zero, because it's f of z.

P. It'd be zero. So,

F. Oh, so it wouldn't be, it wouldn't be differentiable and analytic at a point where z equals zero. No where z equals, no, oh, my goodness. Uh, basically, I guess it's not going to be analytic where you approach the same point from the top and the bottom. Right?

P. Yes.

F. So that would be the real part only. If the imaginary, so it's analytic on all the imaginary, but not on the reals. Right?

P. I think you've got the idea.

F. Does that make sense? I'm trying to think through in my own head.

P. So the imaginary part got to be zero.

F. And the real part, it just can't approach the same point from top to bottom.

P. Right, so where are the points where that's possible?

F. Where are the points? Sorry where what's possible?

P. Where you can approach from both top and bottom?

F. Oh, the reals.

P. The real line?

F. Yeh.

P. It's analytic everywhere else.

F. Except on the real line. Oh, OK.

[tape ends]

Interview #1, Class 3, Spring 1998

1. What is the real part of the following?

a. $\underline{4} + 2i$

b. $3 - (1 + i)i$ $= 3 - i - i^2 = \frac{3+1}{4} - i$

Question 1

P. OK, so we're starting with problem 1, part a).

M. So, I can write on this?

P. Yes.

M. All right. So the real part is 4.

P. OK.

M. And I'd multiply i through in the second part, so I get, 3 minus i, minus i squared, equals 3 and i squared is minus one, so the real part is 4.

P. OK, I just want to back up, in part a, you could just read it off? Is that right? You could just read off the real part?

M. Um, yes.

P. And in part b), why couldn't you just read it off?

M. Um, well xx, there is a real and imaginary part in the second part, OK, I guess, for a complex number, you have, it's real part is simply a real number,

P. OK.

M. And the imaginary part is just a real number multiplied by i,

P. OK.

M. So I can see, that there is going to be a mixture in there.

P. OK.

M. So that's why I evaluate that.

P. OK, so you had to multiply this out. All right.

M. I checked my algebra.

P. that's not the point, the point is that you couldn't just read it off.

M. Oh, no. No it didn't jump out at me that that was 4.

P. Sorry, that's not what I'm after either. The point is that, it looks like you had to change this [(1+i)] back into a real number, is that right? You had to convert? You had to have a real number in front of the i before you could be sure what the real part was?

M. Um, yes.

P. OK.

M. I'm still not clear on your question.

P. OK. Um, let's back up. When you chose to multiply through by i, it was because why? I mean why couldn't you just like the first part, a) where you had something multiplied by i, and something not multiplied by i, why couldn't you just take the real part to be the part not multiplied by i? Three in this case.

M. Well, because there's also a real part in here. I see, the i multiplied by i, which I know will give a real value.

P. OK. So that's how you're thinking about it.

M. Yes.

P. OK. So you just look at it and you see this looks like there a real part there, so you're going to multiply it out.

M. Yes.

P. To find out what that is.

M. Yes.

P. OK.

M. I mean, I don't, a faster way doesn't jump out at me.

P. Oh, that's OK, I ** how fast you go. Question 2.

M. All right.

P. Part a)

---

2. If we use the z = (x, y) representation for complex numbers, which of the following are correct statements? (Circle the correct ones)

a. $3 + 6i = (3, 6)$

b. $3 + 6i = 3i(-i, -2)$

c. $3 + 6i = 3i(-i, -2i)$

d. $3 + 6i = (3, 6i)$

e. $3 + 6i = (2 + i, -i + 5)$

f. $3 + 6i = (1+2i)(1, 2)$

g. $3 + 6i = 3(1, 2)$

h. $3 + 6i = (2 + i, 1 + 5i)$

Question 2

a)

M. ......Um, all right, so I would look at the real part of this, and see that it is the real part of this notation, and I see that the imaginary parts correspond as well. So a) is correct.

P. OK, a) is correct.

b)

M. ...well, I guess I'll go to b) next.

P. Yeh, it would help me on the tape if you did them in the right order. So in part b)

M. So for b), again I would need to multiply it out, expand it to see,

P. OK, go ahead.

M. **...so that would be, so that one is not correct.

P. You got 3 i times,

M. Oh, wait, ...

P. OK.

M. Did I do that right? Yeh, minus 6 i. So that is not correct.

P. OK, it's not correct, because,

M. Because the imaginary parts don't match.

P. OK. They don't match because it's minus 6 instead of 6?

M. Yeh.

c)

M. ....So that looks right.

P. OK, so in part c), again you multiplied the 3 i through?

M. Yes.

P. And that came to (3, 6)? i squared is minus one, OK. All right,

M. So they match, the real and imaginary parts match.

d)

M. all right this is not correct, because the second place in the bracket ** is imaginary, so to put an i in there is incorrect.

P. OK.

M. so that would be the equivalent of, well this wouldn't have an imaginary part. So it's not right. If I just saw that, I would think that that meant 3 minus 6. so that should be completely real.

P. OK. ** [both talking]

M. ** And this is obviously complex.

P. OK. So part e).

e)

M. ....Well, my initial reaction is that it is wrong. I can't see how 2 plus i will equal 3. I don't see how, **...................No, that's not correct.

P. How come?

m. Well, it's not possible to make 3 equal to 2 plus i...............Um, wait a second. 2 + i, +, so I'm simply taking the second part, and multiplying it by i.

P. OK.

M. ....2 + i + 1 + 5i, OK, so that's, so I was wrong this time, it is correct.

P. OK.

M. But it definitely is not a form I've seen before. So the only way to do it is actually by working it out.

P. So what exactly do you think is wrong with it in that form? It's obvious it bothers you, so.

M. Yeh, well for me a complex number should be, in this notation, you should simply have real parts in the brackets.

P. OK.

f)

M. So again I'll have to expand this one, I'll have to multiply out the second part of that.

P. OK.

M. ....All right, I multiply it out..........So that's, they do not seem to be equal. Neither the imaginary or the real parts are the same.

P. OK. **, you wrote (1, 2) equals (1 + 2i) and then just multiplied it.

M. Yes.

P. OK.

g)

M. All right for g), I take one comma, 2 in brackets as 1 plus 2i, and then multiply the three through.

P. Hmhm.

M. That give me 3 plus, 6 i which is correct.

P. OK.

h)

M. And for part h), that's in a similar form to part e), ..so, **, ...actually work out what the right side is, so I take 2 plus i, to be a real part, + 1 plus 5i to be the imaginary part, and multiply that by i,

P. OK.

M. So that will get, ...so, that appears to be wrong. neither the real or the imaginary parts are the same.

P. So you multiplied, no you added 2 plus i, to 1 plus 5i, all times i,

M. Right.

P. And got, i minus 5 [for 1 plus 5i times i], so it worked out to minus 3 plus 2i, which doesn't equal 3 plus 6i, OK.

M. Right.

P. So you reject it.

M. Hmhm.

P. OK, so, um, ...

M. Should I move on to number 3?

P. Um, for the time being let's go on to three.

---

3. a) Put the following into a + bi form.

i) $\frac{2+4i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{2-4i+4i-8i^2}{1+4}$

$= \frac{10}{5} = 2$

ii) $\frac{-6+3i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{-6+12i+3i-6i^2}{1+4}$

$= \frac{15i}{5} = 3i$

Question 3 a)

a i)

M. All right number 3 a), part i), OK, so I see that I need to put it in a simpler form than it already is,.

P. Hmhm.

M. So I see that there is an i in the denominator, and you can get rid of it by multiplying through by the complex conjugate of the denominator.

P. OK.

M. ** multiply by one minus 2i over one minus 2i. **..., so I know the denominator won't **, 1 plus 4, and multiplying by the top, ...this is, 10,...2.

P. OK. So you top and bottom by the conjugate, of the denominator, and worked it out, and it came out to 2.

M. Yes.

a ii)

M. Part 2, so I'll do the same thing as in part one, I'll multiply, the expression by the complex conjugate of the denominator, so I get one minus 2i, over one minus 2i, so one plus 4 in the denominator, ..

P. ***

M. All right, sorry,

P. That's OK.

M. All right, so I multiplied the denominators, and that gives me 5 on the bottom, and multiplying the top through, you get minus six, ...plus 12i, plus 3i, minus 6i squared. Now I combine the real parts, so minus 6, minus 6 i squared, is minus 6, plus 6, zero for the real part, and the imaginary parts give 12i, plus 3i, is 15i, over 5, equals 3i.

P. OK.

M. Makes sense?

---

b.) Three students did the following problem three different ways. Which of them are correct? (Circle the correct ones)

i) $\frac{6-9i}{2-3i}=\frac{6-9i}{2-3i}\cdot\frac{2+3i}{2+3i}=\frac{12+27}{4+9}=\frac{39}{13}=3$

$18i-18i-27i^2$

ii) $\frac{6-9i}{2-3i}=\frac{6-9i}{2-3i}\cdot\frac{2+3i}{2+3i}=\frac{12-11i+9i+27}{4+9}=\frac{39-2i}{13}$

iii) $\frac{6-9i}{2-3i}=\frac{3(2-3i)}{2-3i}=3$

$\frac{a+ib}{a+ib}\cdot\frac{a-ib}{a-ib}=\frac{a^2+b^2}{a^2+b^2}$

Question 3 b i)

M. Part b number one, so I look at the first left most expression, and it is the quotient of two complex numbers, and I check to see if they multiplied by the complex conjugate of the denominator, and it appears they have, and now I check their multiplication. OK, the denominator is correct, ...so I can't do all of it in my head, I'll have to write it out.

P. Go ahead.

M. The real part is right, plus 12, then you have 18 i, minus 18 i, so those cancel, there's no imaginary part, that's right.

P. Hmhm.

M. And we have, minus 27 i squared, equals plus 27, which is right, and, ...**, 39 over 13 is, 3, so that's correct. I'll circle that one,

P. OK, so you just, worked through their conjugation operation and you decided it was correct.

b ii)

M. Part 2, again, you have, the same quotient of complex numbers,

P. Hmhm.

M. I see that they have multiplied by the complex conjugate of the denominator, the denominator gets the same answer **, 13, that's fine.

P. Hmhm.

M. And, I see, this time they have included extra stuff in the, in the multiplication of the complex numbers in the numerator, and, ...I see that they've made a mistake, in multiplying the imaginary parts, ...so it's seems the second part is wrong, in that there should be no imaginary part.

P. OK, do you know,

M. The imaginary parts do not cancel.

P. OK.

M. So they multiplied 6 by 3i, wrong....

P. OK.

M. As well as multiplied 2 by minus 9i wrong.

P. OK.

b iii)

M. All right, so part 3? .....OK, same quotient of complex numbers, ...so I looked at the numerator, and they have factored the numerator, they've taken a 3 out of it,

P. Hmhm.

M. And they've done that correct. So I see that, this 2 minus 3i cancels in the numerator and the denominator, to get the right answer of three. So part three is correct as well.

P. OK, summarizing, in iii)?

M. Summarizing in iii), it was not necessary to multiply by the complex conjugate of the numerator, because we could factor the numerator, in to some, and it canceled the denominator.

P. And that's, that's allowed? To cancel a complex number from top and bottom?

M. ....Um, ...yes. A complex number divided by itself is one.

P. OK, are you sure?

M. No, I'd have to work out a general from a + ib, and then **.

P. OK.

M. Intuitively, I'd say yes,

P. OK, so you,

M. I would need to, I would need to check that, in a general form.

P. OK, you can check it on the side there if you want.

M. All right, so try, a general complex number z, which a plus ib,

P. Hmhm.

M. Over a plus ib, multiply through by the complex conjugate, which is a minus ib, top and bottom,

P. Hmhm.

M. Which gives, in the denominator, a squared plus b squared, and in the numerator it is, a squared plus b squared, so the numerator and the denominator are the same, a squared plus b square, is equal to one, so therefore, a complex number divided by itself is one.

P. You're sure this [a squared plus b squared, over a squared plus b squared] is one how come?

M. Because the numerator and denominator are equal.

P. They're equal what kind of numbers?

M. Real numbers.

P. Real numbers.

M. One. And a real number divided by itself is one [M's proof, of course, uses what we are trying to prove, i.e., when multiplying by a minus ib, over a minus ib, we have to know this is one.].

P. OK, that's good. So now you know that they are correct.

M. So I would say that part three is correct. You can cancel the complex numbers.

---

c. **Put the following into a + bi form.**

i) $\frac{-4+2i}{2-i} \cdot \frac{2+i}{2+i} = \frac{-8 \cancel{-4i} \cancel{+4i} + 2i^2}{4+1} = \frac{-8+2}{5}$

$-2(2-i) \quad = -\frac{10}{5}$ [illegible] -2

ii) $\frac{6+2i}{-3-i} \quad \frac{2(3+i)}{-(3+i)} = -2$

Question 3 c i)

M. So part c) number i). So again we have the quotient of two complex numbers, which I'll put in the form of a single complex number,

P. Hmhm.

M. So again I have to, I choose to get rid of the of the imaginary part of the denominator, so I would multiply through by the complex conjugate of the denominator, which is 2 plus i,

P. OK.

M. And multiplying out, the denominator is 4, um, plus one would be 5, and numerator is minus 8, minus 4i, plus 4i, so the imaginary parts cancel in the numerator, plus 2 i squared equals, minus 8 plus 2 fifths. Minus 2, over 5, which is equal to minus 8 fifths for the real part, minus 2 fifths for the imaginary part. Ah, **!, **, just one second, there is no imaginary part, it is minus 10 over 5, so it's minus 2.

P. Is there an easier way to do that one?

M. Well, you could, factor minus 2 from the numerator and that would make it quicker.

P. What would that look like?

M. So factor out a minus two from the numerator would give you, minus two times, in brackets, 2 minus i, 2 minus i is also what the denominator is, so 2 minus i would cancel leaving you with minus 2, which is what I got the first method.

P. OK.

M. So I didn't look for a way to simplify it. I simply use brute force.

P. OK.

3 c ii)

M. So for the second one, my brain has already been primed to think of simplification,

P. Your brain is what?

M. Primed.

P. Oh, it is primed.

M. Yeh, it's been primed, **, from the previous question. OK, I see that the numerator of part ii) can be simplified, from 6 plus 2i, to 2 times in brackets, um, take out the 2, to 3 plus i, and the denominator as well can be, simplified by pulling out a minus one, to give minus one times in brackets, 3 plus i, the 3 plus i's cancel in the numerator and the denominator, give you minus 2 as the answer.

P. OK.

M. So this time I looked for simplification first.

P. Is this a useful thing to do?

M. In this format it is faster.

P. It's faster.

M. It involves less steps.

P. I'm just looking back on part a), could these be done that way?

M. Well, I see that both part a), number i), and part a), number ii) can be simplified. The numerators can both be simplified. I mean can be rewritten, by pulling out, a factor of two from the numerator in part i), and a factor of, um, 3 from part ii). But we can see from part ii) that won't help.

P. Pardon?

M. In pulling out a 3 from the numerator in part ii), three a) part ii),

P. Hmhm.

M. Will leave you with, something that is not equal to the denominator, so, it won't cancel............................So I see that pulling a factor out of the numerator, 3 a) part i), will help.

P. OK. And it's less clear in part ii)?

M. That's right, yeh, in part ii), *** in my head, I can see that it won't get you anywhere, you will not be able to cancel out anything.

P. OK. All right so we can go on to part 4, problem 4.

4. a) Simplify the following expressions, given that z = (x, y).

i) (3, 2)(1, 4) $= (3+2i)(1+4i)$

$$= 3+12i+2i+8i^2$$

$$= -5+14i$$

ii) $\dfrac{(3,2)}{(1,4)} = \dfrac{(3+2i)}{(1+4i)} \cdot \dfrac{1-4i}{1-4i} = \dfrac{3-12i+2i-8i}{1+46}$

$$= \frac{11-10i}{17} -$$

Question 4 a i)

M. Part a), part i)?

P. Yes.

M. All right here we have two complex number multiplied together in this bracket notation.

P. OK.

M. So the way I will do this is I will write out each of the brackets, in the form a plus ib, to get 3 plus 2i, times, 1 plus 4i, which give 3 plus 12i, plus 2i , plus 8 i squared, equals, 3 minus 8, is minus 5 for the real part, and plus 14i, 14 for the imaginary part.

P. Hmhm.

M. All right part ii)?

P. So just reviewing again, you converted into,

M. I converted from the brackets notation with a comma, to a plus ib form, for each of the complex numbers, and then simply multiplied them out.

P. And you did that for what reason?

M. Um, that seemed to me, as I said, to be the best way to do it. Maybe you can multiply the brackets out, directly, I don't know. I'll look and see.

P. OK.

M. It doesn't feel natural to do it that way. So the way I've done it is the way I would do it.

P. OK, that's good.

M. So part ii)?

P. Yep.

4 a ii)

M. So here we have the quotient of two complex numbers in this bracket comma notation.

P. Hmhm.

M. So I rewrite both of those in a plus ib, ...so 3, 2 will be 3 plus 2i, the denominator will be 1 plus 4i, and I don't see any factors that can be pulled out of either of the numerator or denominator, so I will simply multiply through by the complex conjugate of the denominator, so that will be 1 minus 4i, and multiplying you get a denominator of one plus 4 is five, the numerator will be 3 minus 12 i, plus 2i, minus 8 i squared, which is equal to 11 for the real part, minus 10 i all over 5, ...and that's probably how I would leave it.

P. OK, what's the denominator?

M. Oops, um, that should not be 4 in the denominator, it should be 16, yes, I didn't square 4.

P. Yes, OK.

M. All right that gives a denominator of 17, instead of 5.

P. OK. That's it.

b) Simplify the following given that $z = re^{i\theta}$.

i) $3e^{i\pi} \times 5e^{i\frac{\pi}{2}}$ $= 15e^{i(\pi + \frac{\pi}{2})} = 15e^{i\frac{3\pi}{2}}$

ii) $3e^{i\pi} \div 5e^{i\frac{\pi}{2}}$ $= \frac{3}{5}e^{i(\pi - \frac{\pi}{2})} = \frac{3}{5}e^{i\frac{\pi}{2}}$

iii) $3e^{i\pi} + 5e^{i\frac{\pi}{2}}$ $= 3(\cos\pi + i\sin\pi) + 5(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= 3(-1 + i0) + 5(0 + i1)$

$= -3 + 5i$

Question 4 b i)

M. OK, so number 4 b)?

P. 4 b).

M. Number i), ...OK, so here we have complex numbers in exponent, sorry in polar form, OK, so we have two numbers, complex numbers in polar form multiplied together, so, what I'll do is I'll multiply, the coefficients of these exponentials, 3 times five gives you 15, and add the exponents of exponentials, to get i, the exponent is i times pi plus pi over 2, to get, 3 pi by 2, so it would be 15 times e to the 3 pi by 2 i,

P. OK.

M. ....So part ii)?

P. Yes.

4 b ii)

M. All right, now the quotient of two complex numbers in polar form, I see that I divide the coefficients now, of the polar numbers, and that 3 fifths, and I subtract the exponents, so 3 fifths, e to the i pi minus pi over 2, which is pi over two, so it's 3 fifths e to the i pi over 2.

P. OK.

4 b iii)

M. And, in the third part, the, we have two complex numbers in polar form, but they are added together, so we can't operate on it directly, I would convert this into, um, I would convert it into complex numbers, of form r, theta.

P. OK, go ahead and do it that way.

M. ....the left side is 3 cos of pi, plus sin of pi, and the right complex number is 5 and cos of pi by two plus sin of pi by 2, ...all right, **, 3 times cos pi is minus 1, plus sine pi is zero, plus 5 times cos of pi by 2, is 0, and sin pi by 2 is 1, so that gives us, multiplying the coefficients, minus three plus five i,

P. OK.

M. ....OK, I guess I could have done this differently too, I could have recognized the e to the i pi, as minus one,

P. OK.

M. And that e to the i pi over 2, is e to the i pi, er, to the, to the half, I square root it, so that would be the square root of minus one, so, that would be i. So that would be an alternative way to do it. I saw that second.

P. OK. Let's go on to problem 5.

5. a) **Mark the following six statements about complex numbers as true or false.**

T 1) **A complex number is a point in the complex plane.**

T 2) **A complex number consists of a real part, a, and an imaginary part, b, that are added together (with a factor of i in front of b) to get z = a + ib.**

T 3) **A complex number is the endpoint of an arrow that starts at the origin of the complex plane.**

T 4) **A complex number is an ordered pair, such as, z = (x, y).**

T 5) **A complex number is a vector in the complex plane.**

T 6) **If Rez is x, and Imz is y, the z = x + iy.**

Question 5 a)

1)

M. Part a), all right, so a complex number is a point in the complex plane. True.

2)

M. ....A complex number consists of a real part, a, and an imaginary part, b, with a factor of i in front of b, to get z = a + ib. True.

3)

M. A complex number is the endpoint of an arrow that starts at the origin of the complex plane.......I guess if you represent complex numbers as vectors, that would be true. If you took them to be vectors, so I would say true.

P. OK.

4)

M. A complex number is an ordered pair, such as z = (x, y). True.

5)

M. A complex number is a vector in the complex plane. Well, from 3 above, if 3 is true then, 5 will be true.

P. This 5 **, just independently is 5, do you like 5?

M. ....Um, I'd say I like it as much as 3, given that I've seen 3 **, I'm more likely to accept 5.

P. OK.

6)

M. ......If the Rez is x and Imz is y, then z = x + iy. True. xx five a) is true. Part b)?

P. OK.

---

b) Which of the above six statements best describes how you think of a complex number? (Circle one) 1 2 3 4 5 (6)

Question 5 b)

M. All right so, I think of a complex number z where there is a component real, plus s an imaginary component, so I think of it mostly as parts, where z is x plus i y,

P. OK.

---

c) **Describe how you think about complex number in you words.**

A complex # is an ordered pair in a plane with the horizontal component being Re z and vertical component is the Im z.

Question 5 c)

M. ....Part c). How do I think about complex numbers. I think of them as, **, as an order paired, except you define the vertical axis to be imaginary, so, and you, then you, how would you say, vertical value, represented by that value times i,

P. OK, you can write that down.

M. So I would say, a complex number is an ordered pair in a plane with the horizontal component being the real part of z, the complex number, and the vertical component is the imaginary part of z.

P. OK.

M. Number six?

P. Can I go back here a minute?

M. Sure.

P. A complex number is an ordered pair in the plane, with the horizontal component being the real part of z, and a vertical component being the imaginary part of z,

M. So horizontal, being the x component, and the vertical being the y component.

P. OK.

M. So looking at it, it's just a regular, point, in R two,

P. OK.

M. Where you have x and y.

P. OK.

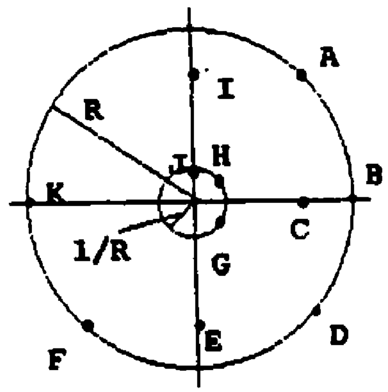
M. x is **, y is not.

P. OK.

M. Next one?

---

6. Use the diagram to fill in the blanks below. Select a point for each item. The large circle has radius R, and the small cirlce has radius 1/R. The first one is done for you (and tells you where z is).



i) z is __A__

ii) |z| is __R__

iii) z bar is __D__

iv) 1/z is __G__

v) Im (z bar) is __E__

vi) Re z is __C__

vii) |z bar| is __R__ B

ix) 1/(z bar) is __A__

x) Im z is __I__

xi) Re (z bar) is __C__

xii) -z is __F__

$$\frac{1}{\bar{z}} = \frac{1}{x-iy} = \frac{1}{re^{-i\theta}} = \frac{1}{r}e^{i\theta}$$

Question 6

M. Number six. Use the diagram to fill in the blanks below, select a point for each item. The large circle has a radius R, and the small circle has a radius of 1 over R. The first one is done for you, and tells you where z is. So I'll look at number i), z is point A.

P. Hmhm.

ii)

M. It lies on the perimeter of the outer circle, all right, so, um the modulus of z is, the distance, or I should say it's just the radius of the circle, so R. I don't see a specific point, so it's just R.

P. ***, OK.

iii)

M. z bar, the complex conjugate of z, would be a flip, in the, in the real axis, or a reflection over the real axis, I should say. So the corresponding point to A, is D.

iv)

M. one over z, ...is, ...one over the magnitude of z, so that would be one over R, and the polar angle will be the negative of the polar angle for z.

P. Hmhm.

M. So it should be in line with point D, but on the inner circle of radius one over R, so it should be point G.

P. OK.

v)

M. So the imaginary part of z bar, that's the imaginary part of complex conjugate of z, ...OK, so, ...sorry, my mind has gone blank. OK, I'm now thinking about the question again. OK, z bar, is point D, the imaginary part of point D, is point E.

vi)

M. All right, the real part of z, is point C.

vii)

M. The modulus of z bar should be the same as the modulus of z, which is R.

P. So if there's a point on there to represent this [the modulus of z bar], then what would be the best point?

M. OK, so we pick z is A, z bar is D, ...the modulus of z bar, ...well the modulus is a scalar, oh, so I guess point B is purely real.

P. OK.

M. It's going to be at the same distance to the origin as point B.

P. OK. And why,

M. It's going to be real.

P. OK, why isn't it K?

M. Because the modulus is not negative.

P. OK.

ix)

M. All right, one over z bar, again z bar has the same modulus as z, so the modulus of one over z bar equals the modulus of one over z, which is one over R, so it will be on the inner circle........and, OK, z bar is point D, so ** take the negative of that angle will bring you back to point A, which will be ***.

P. Sorry, one over z bar is, can you go through that reasoning again?

M. I'm actually going to write it all on paper.

P. OK.

M. All right, so I'll write z in x plus i y form, so you have one over z bar, which is one over x minus i y,

P. Hmhm.

M. But that's not **, so I'll write it in polar form. So that would be one over, I, modulus of z, e to the minus i theta, so that brings you, oh, one over z bar is one over, r times e to the minus i times theta, and one over an exponential that is negative is just an exponential, so you invert it again, and that would give you one over r, e to the i theta,

P. Hmhm.

M. All right, so it has the same argument as z, [tape change]

P. OK.

M. So again, one over z bar should be on the inner circle, the modulus is one over the modulus of z,

P. Hmhm.

M. But it has the same argument as z. So it should be point H.

P. Oops, that was one over z bar.

M. Which, ..OK. So for part ix) is H.

P. OK.

x)

M. Ah, for the next one, the imaginary part of z, z is point A, the imaginary point is I.

P. Why is that?

M. You're simply looking at the vertical component of A.

P. OK.

M. xx, I don't know any better way to explain it.

P. OK, are you saying that the imaginary part is real or imaginary?

M. ....Yeh, it is a real value, but in the complex direction.

P. OK. The complex direction being pure imaginary?

M. Yes.

P. so you're saying the imaginary part of z, includes the i then?

M. So, yeh, it's, the imaginary part of z is, the magnitude of the vertical part, in units of i.

P. Times i?

M. Times i.

P. OK.

M. Yeh.

xi)

M. Part eleven, the real part of z bar, so you know z bar is point D, and the real part of that is the horizontal component, of point D, is just point C. It's the point on the real axis, which corresponds to point D.

P. OK.

M. I guess that is a better way of explaining the answer. So point C on the real axis.

xii)

M. Number 12, negative z, ...would mean you are simply taking the negative part of the magnitude, so that should be point F...........Yes.

P. OK.

M. Yeh, that's the negative of both the horizontal and vertical components.

P. OK. All right, unless you want to go on you can stop.

M. Oh, is it that time? I can keep going if you like.

P. OK. Try this one.

M. For five more minutes?

P. Yeh. OK.

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7. a) Find the solutions of the following equations:

i) $z^2 + 2iz - 1 = 0$

$$z = \frac{-2i \pm \sqrt{4i^2 - 4 \cdot 1 \cdot (-1)}}{2}$$

$$= \frac{-2i \pm \sqrt{-4 + 4}}{2}$$

$$(z+i)^2$$

$$= -\frac{2i}{2} = -i$$

Question 7 a)

M. Find the solutions of the following equations. z squared plus 2iz minus one equals zero. ....So my reaction is to apply the quadratic equation directly to it.

P. OK.

M. So we'll do that.

P. See what happens.

M. z equals negative of b which equals minus 2 i, plus or minus the square root of b squared which is 4 i squared, minus 4 times one, times c which is minus one, all over 2 a which is 2 times one. Simplifying, that's, minus 2 i plus or minus, the square root of minus 4 plus 4, so the square root is zero, over two, so we have minus 2 i over 2, equals, minus i.

P. OK, so first of all, the quadratic formula still works? It's a valid way of doing it?

M. I can't say for sure, no, I'm not 100% comfortable. And probably on the test I would work through it with z expressed in x plus i y form, and see if the algebra still holds.

P. OK, so is it OK that we only have one root?

M. Um, ...let me see if, ...I'm not sure if we can factor this like you can factor a real polynomial, but I assume so, I'll see what happens.

P. OK, let's see what happens.

M. So ** z minus i, squared, that's z minus i, times z minus i, which would come down and give you one root as well, i......Did I do this wrong, oh, sorry, it should be z plus i squared. OK, so that would give you a root of minus i, which is consistent with **.

P. OK, so, do you, have any more confidence in the quadratic formula now?

M. I have more confidence that it works, I get the same results two different ways.

P. OK. Why don't we stop. Thank-you very much.

End of Tape